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19. ABSTRACT (Continue on reverse if necessary and identify by block number)  A numerical procedure is given for evaluating $P$ , the integral of the trivariate normal distribution over an offset sphere of radius $R$ . A numerical algorithm is also described for computing $R$ when $P$ is given. The procedures generally give $P$ or $R$ to six decimal digits. Program listings in HP BASIC, for the HP 9845B computer, are included.  <i>Keywords: trivariate normal distribution, offset sphere, numerical algorithm, HP BASIC, HP 9845B computer.</i>				
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11. INVERSE PROBLEM

## FOREWORD

The work described in this report was done in the Space and Surface Systems Division of the Strategic Systems Department. Its purpose was to design software, suitable for a high quality mathematics and/or statistical subroutine library, which yields probabilities and percentage points for 3-dimensional normal distributions over off-set spheres.

The author is grateful to Mel Holloway for his guidance and direct assistance with the technical word processor that was used to produce the draft copy of this report.

Released by:

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## I. INTRODUCTION

In the application of the methods of operation research to problems of military strategy, it is sometimes necessary to compute the value of a kill probability expressible in the form

$$P = \iiint_{-\infty}^{\infty} p(m) F(x,y,z,u,v,w) dx dy dz, \quad (1)$$

where

$$\begin{cases} F(x,y,z,u,v,w) \equiv \frac{1}{2\pi \sqrt{2\pi} uvw} E(x/u) E(y/v) E(z/w) \\ E(t) \equiv \exp(-t^2/2). \end{cases} \quad (2)$$

Here  $P$  is the kill probability. The shots or bursts are assumed to form an uncorrelated trivariate normal distribution with their mean or expected burst point set at the origin of a rectangular cartesian coordinate system  $Oxyz$ . The distribution has standard deviations  $u, v, w$  in the  $x, y, z$  directions, respectively. This configuration may be assumed without loss of generality for any trivariate normal distribution, since if the distribution is correlated and does not have its mean at the origin, a translation and a rotation of axes will bring about the situation which has been described.

A target  $T$ , which may be either a point or area target, is assumed in some arbitrary position in the  $Oxyz$  space. The distance  $m$  is the radial or slant range miss distance of an arbitrary burst point  $(x,y,z)$  from the target. If a target is a point at a position  $(H,K,L)$ , then  $m$  is the ordinary distance  $[(x-H)^2 + (y-K)^2 + (z-L)^2]^{1/2}$ . If  $T$  is an area target,  $m$  is the minimum distance from  $(x,y,z)$  to points of the target. For area targets  $m$  is set to zero for all burst points  $(x,y,z)$  lying within the target.

The expression  $p(m)$  is a function of  $m$  as prescribed on the basis of physical conditions or assumptions of the problem. Possible forms of this function which have practical applicability are: (1)  $p(m) = 1$  if  $m \leq R$  (a constant),  $p(m) = 0$  otherwise; (2)  $p(m)$  is a decreasing linear or exponential function of  $m$ ; (3)  $p(m)$  is a monotonically decreasing step function.

This report describes a procedure for the efficient computation of these kill probabilities by a high speed digital computer for an important practical case, that in which the target is a point at an arbitrary position  $(H,K,L)$  where, without loss of generality,  $H, K, L$  are nonnegative and the function  $p(m)$  has the first form mentioned above, i.e.,  $p(m) = 0$  for  $m > R$ ,  $p(m) = 1$  for  $m \leq R$ . The constant  $R$  may be thought of as the lethal radius of the weapon.

The function  $p(m)$  has the effect of reducing the field of integration to the interior of the sphere  $S$  of radius  $R$  with the target  $(H,K,L)$  as center, i.e.,

$$S : (x - H)^2 + (y - K)^2 + (z - L)^2 = R^2. \quad (3)$$

The kill probability  $P$  is then given by

$$P = \int_{L-R}^{L+R} \int_{K-Y}^{K+Y} \int_{H-X}^{H+X} F(x,y,z,u,v,w) dx dy dz, \quad (4)$$

where

$$X = \sqrt{R^2 - (y - K)^2 - (z - L)^2}, \quad Y = \sqrt{R^2 - (z - L)^2}. \quad (5)$$

Rather than carry out the numerical triple integration of Equation 4 directly we make use of an available computer program that yields the probability  $P_C(\bar{r}, H, K, u, v)$  of a shot falling under a bivariate normal distribution inside a circle in the Oxy-plane of radius  $\bar{r}$  and center  $(H, K)$ .  $P_C$  is the two-dimensional analog of  $P$ .<sup>4,5,6</sup>

Geometrically, one observes then that  $P$  can be obtained by considering circular slices of  $S$  parallel to the Oxy-plane. For a fixed  $z$  in  $[L - R, L + R]$ , the xy-integration in Equation 4 over a slice of radius  $\bar{r} = \sqrt{R^2 - (z - L)^2}$  yields  $P_C$ . Weighting  $P_C$  with  $1/(\sqrt{2\pi} w) E(z/w)$  and integrating the result over  $z$  in  $[L - R, L + R]$  gives  $P$ , i.e.,

$$P = \frac{1}{\sqrt{2\pi} w} \int_{L-R}^{L+R} E(z/w) P_C(\bar{r}, H, K, u, v) dz, \quad (6)$$

where

$$\bar{r} = \sqrt{R^2 - (z - L)^2}, \quad (7)$$

or

$$P = \frac{1}{\sqrt{\pi}} \int_{(L-R)/\sqrt{2} w}^{(L+R)/\sqrt{2} w} E(\sqrt{2} t) P_c(r, H, K, u, v) dt, \quad (8)$$

[From (2),  $E(\sqrt{2} t) = \exp(-t^2)$ ]

$$r = \sqrt{R^2 - (L - \sqrt{2} w t)^2}. \quad (9)$$

Hereafter, we assume  $P_c$  is available and the evaluation of  $P$  reduces to the single integration indicated in Equation 6 or Equation 8.

The symmetry properties of  $F$  (see Equation 2) allow  $H, K, L$  to always be taken as nonnegative. Since  $F > 0$  and bounded, the order of integration in Equation 4 is immaterial. Thus, as long as  $H$  is associated with  $u$ ,  $K$  with  $v$ , and  $L$  with  $w$ , it does not matter thereafter which is called  $L$  and  $w$ . For example, if the order of integration is chosen so that the  $x$ -integration is performed last, then if initially  $H = 10, u = 5, L = 20, w = 7$ , we simply let  $H = 20, u = 7, L = 10, w = 5$ . In this way, we can always refer to Equation 6 or Equation 8 as the basic representation for  $P$ , where  $L$  and  $w$  are always associated with the  $z$ - or  $t$ -integration indicated in Equation 6 or Equation 8 with the understanding that the order of integration may have been changed and the variables  $H, K, L$  and  $u, v, w$  renamed as in the example above.

## II. COMPUTATION OF $P$

Sometimes we shall write  $P(R)$  or  $P(R, H, K, L, u, v, w)$  for  $P$ . As stated in the previous section,  $R$  denotes the radius of the sphere  $S$  given by Equation 3. The trivariate normal distribution  $F$  is defined in Equation 2 by its mean  $(0, 0, 0)$  and its standard deviations  $u, v, w$  in the  $x, y$ , and  $z$  directions, respectively. By appropriate normalization of the variables  $R, H, K, L, u, v, w$  it is easy to show that  $P$  is in general a function of six independent variables. The program for computing  $P$  is called ELLCOV.

In most cases  $P$  will be computed by the numerical integration of Equation 8, where it is assumed  $P_c$  is available. There are two special cases however where  $P$  can be evaluated in closed form. We shall call these cases A and B. The derivations of the results for these will be given in Appendix A. Case A is found in



the literature<sup>10</sup>. Case B is also probably given in the literature, but we have not found it.

CASE A:  $\boxed{u = v = w}$   $D \equiv \sqrt{H^2 + K^2 + L^2}$

$$P = \frac{1}{2} \left\{ \operatorname{erf} \left( \frac{D+R}{\sqrt{2}w} \right) - \operatorname{erf} \left( \frac{D-R}{\sqrt{2}w} \right) \right\} - \sqrt{\frac{2}{\pi}} \frac{R}{w} \left[ \frac{\exp(d) - 1}{d} \right] E \left( \frac{D-R}{w} \right), \quad (10)$$

$\underline{D \neq 0}$

$$P = \operatorname{erf} (R/\sqrt{2}w) - \sqrt{\frac{2}{\pi}} \frac{R}{w} E(R/w), \quad \underline{D = 0}, \quad (11)$$

where

$$\begin{cases} d = -2DR/w^2 \\ \operatorname{erf} x \equiv \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) dt. \end{cases} \quad (12)$$

CASE B:  $\boxed{u = v \text{ and } H = K = 0.}$

The two situations  $u > w$  and  $u < w$  are treated separately.

$u > w$

$$P = \frac{1}{2} \left\{ \operatorname{erf} L_4 - \operatorname{erf} L_5 - \frac{1}{L_2} E(R/u) \exp \left\{ .5 [L/(L_2 u)]^2 \right\} \right. \\ \left. \cdot (\operatorname{erf} F_1 - \operatorname{erf} S_0) \right\}, \quad \underline{S_0 < 0} \quad (13)$$

$$P = \frac{1}{2} \left\{ \operatorname{erf} L_4 - \operatorname{erf} L_5 - \frac{1}{L_2} \exp(-L_5^2) [\exp(S_0^2) \operatorname{erfc} S_0 \right. \\ \left. - \exp(-2RL/w^2) \exp(F_1^2) \operatorname{erfc} F_1] \right\}, \quad \underline{S_0 > 0} \quad (14)$$

$$P = \operatorname{erf} L_4 - \frac{1}{L_2} E(R/u) \operatorname{erf} (RL_2/\sqrt{2}w), \quad \underline{L = 0} \quad (15)$$

$u < w$

$$P = \frac{1}{2} \left\{ \operatorname{erf} L_4 - \operatorname{erf} L_5 - \frac{2}{\sqrt{\pi} L_2} \exp(-L_5^2) \left[ \operatorname{daws}(F_1) - \exp \left( -\frac{2RL}{w^2} \right) \operatorname{daws}(S_0) \right] \right\} \quad (16)$$

$$P = \operatorname{erf} L_4 - 2/(\sqrt{\pi} L_2) E(R/w) \operatorname{daws}(RL_2/\sqrt{2}w), \quad \underline{L = 0}, \quad (17)$$

where

$$\left\{ \begin{array}{l} \text{daws } x \equiv E(\sqrt{2}t) \int_0^x \exp(t^2) dt \quad (\text{Dawson's integral [1; p. 298], [3]}) \\ \text{erfc } x \equiv 2/\sqrt{\pi} \int_x^\infty \exp(-t^2) dt = 1 - \text{erf } x, \end{array} \right. \quad (18)$$

$$L_2 = \sqrt{|1 - (w/u)^2|} \quad (19.1)$$

$$L_4 = (L + R)/(\sqrt{2}w) \quad (19.2)$$

$$L_5 = (L - R)/(\sqrt{2}w) \quad (19.3)$$

$$S_0 = L/(\sqrt{2}wL_2) - RL_2/(\sqrt{2}w) \quad (19.4)$$

$$F_1 = L/(\sqrt{2}wL_2) + RL_2/(\sqrt{2}w). \quad (19.5)$$

In order to achieve improved accuracy, Equations 10-19, are used in modified form in the computing program, ELLCOV. It is also not difficult to show that Equations 13, 14, 16 approach Equation 10 as  $u \rightarrow w$ .

If cases A and B do not apply an attempt is made to improve efficiency by sensing if  $P_c$  is numerically constant over the effective range of the  $t$ -integration in Equation 8. Call this case C. We discuss this case below after specifying the effective range of the  $t$ -integration, (see Equation 29).

In general, excluding cases A, B and C,  $P$  is computed by numerical integration. The 24-point Gaussian quadrature rule [1, p. 916] is used for this purpose with, in some cases, an additional 24-point Gaussian.

The most time consuming part of computing the integrand in Equation 8 is the evaluation of  $P_c$ , the elliptic coverage function<sup>4,5</sup>. In most cases  $P_c$  is obtained from the subprogram function ELLCOV. In the special situations that  $u = v$  or  $H = K = 0$ , (referring to Equation 8)  $P_c$  is called the circular coverage function, CIRCVC, and the generalized circular error function, GCEF, respectively<sup>4,5,6</sup>. Both are computed by the subprogram CIRCVC. In these special situations, a parameter  $V$  is set to 1 and 0, respectively, otherwise  $V = 2$ . When  $V = 0$  or 1,  $P_c$  can be computed about 10 times faster than the case when  $V = 2$ . Hence, since

the order of integration in Equation 8 can be changed, the appropriate interchanges are always made to make  $V = 1$ , if possible. If this cannot be done ( $u \neq v$ ,  $v \neq w$ ,  $w \neq u$ ) then an attempt is made to obtain  $V = 0$ . However some instability can appear in this case, when computing GCEF, if

$$1(-5) > u/v > 1(5). \quad (1(N) \equiv 10^N) \quad (20)$$

Hence if  $V = 0$  and Inequality 20 holds then the computation of  $P_c$  is carried out as described below for the case  $V = 2$ .

If neither  $V = 1$  nor  $V = 0$  can be brought about then the order of integration is determined by finding the largest ratio  $a/b < 1$ , where  $a$ ,  $b$  take the values of  $u$ ,  $v$ ,  $w$ . The pair from  $u$ ,  $v$ ,  $w$  giving the largest ratio ( $< 1$ ) is then used in the first two integrations (the evaluation of  $P_c$ ) with the corresponding  $H$ ,  $K$ ,  $L$  values. For example if  $u = 2$ ,  $H = 0$ ,  $v = 5$ ,  $K = 2$ ,  $w = 3$ ,  $L = 6$ , then the  $a/b$  ratios are  $2/5$ ,  $3/5$ ,  $2/3$ . Hence  $v$  and  $w$ ,  $K$  and  $L$ , are interchanged, the integrand in Equation 8 becomes

$$\exp(\sqrt{2}t) P_c(r, 0, 6, 2, 3), \quad (21)$$

where

$$r = \sqrt{R^2 - (2 - \sqrt{2}5t)^2}; \quad (22)$$

the limits of integration become  $(2 - R)/5\sqrt{2}$ ,  $(2 + R)/5\sqrt{2}$ . If the two largest ratios are equal, then the one with the largest denominator is used to fix the first two integrations.

Choosing the integration order in this way permits the most accurate computation of  $P_c$ . This remark is based on an empirical study in Reference 4, where it was noted that the numerical quadrature of  $P_c$  is best when the standard deviations are nearly equal and not small. We mention at this point that the program ELLCV has been up-graded over its description in Reference 4. Namely, the accuracy for  $P_c$  has been increased from 6 to approximately 8-decimal digits. This was required in order to deal effectively with the inverse problem discussed in Section III.

It is possible, in some cases, by the tests given below to determine if  $P \leq 2.5(-8)$  or  $P \geq 1 - 2.5(-8)$ . In such situations  $P$  is set to zero and one, respectively. The tests below are discussed in Appendix B.

Let  $Z_4 = 2.5(-8)$ ,  $M \equiv \max(u, v, w)$ .

Test #1:

P set to 0 if  $R^3 \leq 1.5\sqrt{2\pi} uvwZ_4$

Test #2:

P set to 0 if  $R^2 < D^2$  and  $R^3 \exp(-\alpha^2) \leq 1.5\sqrt{2\pi} uvwZ_4$ ,  
 $\alpha = (D - R)/(\sqrt{2}M)$ ,  $D = \sqrt{H^2 + K^2 + L^2}$

Test #3:

P set to 0 if  $h - R > \sqrt{2}TF_2$ , where  $F_2 = 4.262$ ,  $h = H, K, L$ ,  
 and T equals the corresponding u, v, or w.

Test #4:

P set to 1 if  $\bar{P} = P(R, H, K, L, M, M, M) \geq 1 - Z_4$ , where  
 $\bar{P}$  is computed from Equation 10 or Equation 11.

For efficiency it is important to try to reduce the number of times  $P_c$  is called by ELLCOV. We note that  $\bar{r}$  in Equation 7 has the same value for two values of  $z$ , i.e.,  $z = L \pm (R - \gamma)$ , where  $0 \leq \gamma < R$ . This observation in many cases reduces by half the number of times  $P_c$  needs to be computed. Indeed, we express Equation 6 as the sum of two integrals, the first has the limits of integration  $L - R$  to  $L$  and the second  $L$  to  $L + R$ . In the second integral, the substitution  $Z = 2L - z$  is made with the following result

$$P = \frac{1}{\sqrt{2\pi}w} \int_{L-R}^L [E(z/w) + E((2L - z)/w)] P_c(\bar{r}, H, K, u, v) dz, \quad (23)$$

or, in terms of Equation 8,

$$P = \frac{1}{\sqrt{\pi}} \int_{A5}^{B5} \{ \exp(-t^2) + \exp[-(\sqrt{2}L/w - t)^2] \} P_c(r, H, K, u, v) dt, \quad (24)$$

where

$$\begin{cases} r = \sqrt{R^2 - (L - \sqrt{2}wt)^2} \\ A5 = (L-R)/(\sqrt{2}w) \\ B5 = L/(\sqrt{2}w). \end{cases} \quad (25)$$

The number of Gaussian abscissae used to integrate Equation 24 is fixed at 24, and in some cases at 48. Greater accuracy may be obtained if the total integration interval,  $B5 - A5 (= R/(\sqrt{2}w))$ , can be reduced. Indeed, consider the infinite volume  $V_e$  bounded by the planes  $|z| = aw$ , where  $a$  is uniquely defined by

$$1/(2\pi\sqrt{2}uvw) \int_{-aw}^{aw} E(z/w) dz \int_{-\infty}^{\infty} E(y/v) dy \int_{-\infty}^{\infty} E(x/u) dx = 1 - \epsilon, \quad (26)$$

$\epsilon > 0,$

or

$$\text{erfc}(a/\sqrt{2}) = \epsilon. \quad (27)$$

The entire region above the plane  $z = aw$  contains  $\epsilon/2$  of the distribution  $F$  given in Equation 2. Some typical values for  $a$  as a function of  $\epsilon$  are

$$\begin{cases} \epsilon = 5(-8) & a \approx 3.8545\sqrt{2} \approx 5.45109 \\ \epsilon = 5/3(-9) & a \approx 4.262\sqrt{2} \approx 6.02738. \end{cases} \quad (28)$$

In ELLCOV we use the latter. The integration over the sphere  $S$ , as indicated in Equation 6, therefore can be reduced to the intersection of  $S$  with  $V_e$  ( $S \cap V_e$ ). In many cases this permits the  $z$  (or  $t$ ) interval of integration in Equation 6 (or Equation 24) to be shortened. The final results for Equation 24 are summarized by

$$\begin{cases} B5 = \begin{cases} L/\sqrt{2}w & \text{if } L \leq aw \\ a/\sqrt{2} & \text{if } L > aw \end{cases} \\ A5 = \begin{cases} -a/\sqrt{2} & \text{if } L-R < -aw \\ (L-R)/\sqrt{2}w & \text{if } |L-R| \leq aw \end{cases} \end{cases} \quad (29)$$

$$W9 = \begin{cases} 0 & \text{if } L < aw \\ 1 & \text{otherwise.} \end{cases} \quad (30)$$

(Note: P set to 0 if  $L - R \geq aw$ .--Reduces to Test #3,  $a = \sqrt{2}$  F2)

The parameter W9 is used in ELLCOV to indicate if the second exponential in Equation 24 can be ignored. The case  $W9 = 1$  implies the exponential can be dropped and  $W9 = 0$  implies it must be retained. The case for  $W9 = 1$  occurs when the integration from  $L$  to  $L + R$  in Equation 6 is negligible since the center of  $S$  lies outside  $S \cap V_e$ , i.e.,  $L \geq aw$ .

A further effort is made to reduce B5-A5 by attempting to increase A5. Let the integrand in Equation 24 be denoted by  $G(t)$  and the increasing Gaussian abscissae by  $t_j$ ,  $j = 1, 2, \dots, 24$ . If  $G(A5) \leq 1(-10)$  then  $G(t_j)$  is evaluated for increasing  $j$  until  $G(t_n) > 1(-10)$ . Then A5 is set to  $t_{n-1}$  and new  $t_j$  are generated based on the new A5.

Since  $G(t)$  has an infinite slope at  $t = (L - R)/(\sqrt{2}w)$ , it may happen that accuracy is lost when  $G(A5)$  and  $G(t_1)$  differ significantly. Once A5 is determined, as described above, if  $G(t_1) > 1(-7)$  a 24-point Gaussian integration is carried out from A5 to  $t_1$ , followed by another 24-point Gaussian integration from  $t_1$  to B5. In most cases  $G(t_1) < 1(-7)$ , in which case only the 24-point Gaussian integration from A5 to B5 is used.

Another sensing to expedite the computation determines when  $P_c$  can be set to one from some  $\bar{t}$  to B5. Using the procedure given in Reference 4 for testing whether  $P_c \geq 1 - \epsilon$ , we have  $P_c \geq 1 - 1.3(-8)$  if

$$r = [R^2 - (L - \sqrt{2}wt)^2]^{1/2} \geq \sqrt{H^2 + K^2} + bs \equiv G, \quad |\sqrt{2}wt - L| < R, \quad (31)$$

where

$$s = \max(u, v), \quad b = 6.02738.$$

Therefore,

$$|L - \sqrt{2}wt| < (R^2 - G^2)^{1/2} \equiv U,$$

or

$$(L - U)/(\sqrt{2}w) < t < (L + U)/(\sqrt{2}w), \quad \bar{t} = (L - U)/\sqrt{2}w. \quad (32)$$

Again, the interval of integration over which the Gaussian quadrature is needed can be reduced when Relation 31 holds and Equation 24 becomes

$$P = 1/\sqrt{\pi} \int_{A_5}^{\bar{t}} \{ E(\sqrt{2}t) + E[(2L - \sqrt{2}wt)/w] \} P_c(r, H, K, u, v) dt \\ + 1/\sqrt{\pi} \int_{(L-U)/\sqrt{2}w}^{(L+U)/\sqrt{2}w} E(\sqrt{2}t) dt. \quad (33)$$

We also note that

$$1/\sqrt{\pi} \int_{(L-U)/\sqrt{2}w}^{(L+U)/\sqrt{2}w} \exp(-t^2) dt = \\ \begin{cases} 1/2 \operatorname{erfc}[(L - U)/\sqrt{2}w] & \text{if } (L + U)/\sqrt{2}w > F2 \\ 1/2 \{ \operatorname{erf}[(L + U)/\sqrt{2}w] - \operatorname{erf}[(L - U)/\sqrt{2}w] \}, & \text{otherwise.} \end{cases}$$

Now that the effective range of integration for Equation 8 has been defined by Equation 29 we can return to case C. If  $A_5 < -F2$  then it is possible for  $P_c$  to remain essentially constant over the effective range of integration.

Let  $r(t) \equiv \sqrt{R^2 - (L - \sqrt{2}wt)^2}$ . Then since  $r(t)$  is an increasing function of  $t$ , we take  $P_c$  as a constant if

$$|r(A_5) - r(B_5)| < 1(-8)r(A_5), \quad A_5 < -F2. \quad (-4.262)$$

In this case  $P$  is given by

$$P = \operatorname{erf}(F2) P_c \{ .5[r(A_5) + r(B_5)], H, K, u, v \}.$$

After much experimentation which included an extensive look at Simpson's rule, the 24-point Gaussian integration scheme was chosen as the basic quadrature procedure. For small  $P$ , a lower Gaussian procedure would have sufficed, nevertheless it was decided for simplicity to use the 24-point Gaussian throughout. The integrating routine is called GQUAD. As a matter of interest, GQUAD allows for the interval  $[A_5, B_5]$  to be subdivided into a set of equal subintervals with the same Gaussian order formula 6, 8, 12, 16, 20, or 24 applied to each subinterval. This feature of GQUAD however is not used in ELLCOV.

Extensive testing of ELLCOV showed that P can be obtained with approximately 8-decimal digits, even though we conservatively claim 6-decimal digits. There does not appear to be any significant limitations, for realistic values of the normalized off-set distance  $\sqrt{(H/u)^2 + (K/v)^2 + (L/w)^2}$ , and the quantities  $u/v$ ,  $v/w$ ,  $R/w$  can have values as large as  $10^5$ .

Below, Table 1 contains some numerical results from ELLCOV. It gives values of P to compare with approximations given previously by F.E. Grubbs.<sup>8,9</sup> A listing of ELLCOV in HP-BASIC is given in Appendix C.



TABLE 1

		R = 1		K = L = 0	
		$\sigma^2 = 1/2$	$\sigma^2 = 1$	$\sigma^2 = 2$	$\sigma^2 = 3$
H = 0	WH	.890	$u = v = w$ .607	.313	.204
	P	.888390	.608375	.317730	.198748
	MC	.885	.615	.320	.210
H = .5	WH	.807	.530	.307	.205
	P	.736060	.501231	.277180	.179569
	MC	.755	.505	.315	.195
H = 1	WH	.337	.261	.174	.127
	P	.337133	.269975	.183329	.132299
	MC	.353	.320	.205	.135
H = 2	WH	.000	.001	.004	.006
	P	3.0986(-3)	1.59356(-2)	3.3474(-3)	3.85359(-2)
	MC	.000	.020	.035	.040
H = 3	WH	.000	.000	.000	.000
	P	1.48(-7)	7.569(-5)	1.7473(-3)	4.7661(-3)
	MC	.000	.000	.000	.010
H = 0	WH	.868	$u = 2v = 2w$ .630	.388	.279
	P	.876323	.647542	.375392	.246952
	MC	.860	.670	.400	.250
H = .5	WH	.739	.554	.364	.269
	P	.747518	.569818	.346921	.233510
	MC	.770	.555	.400	.285
H = 1	WH	.432	.372	.287	.233
	P	.437088	.385964	.273711	.197402
	MC	.435	.405	.290	.235
H = 2	WH	.019	.055	.092	.104
	P	3.08781(-2)	7.57294(-2)	.105474	.100705
	MC	.040	.090	.125	.105
H = 3	WH	.000	.001	.011	.023
	P	1.679(-4)	4.2614(-3)	2.11601(-2)	3.26805(-2)
	MC	.000	.000	.035	.050

$$\sigma^2 = u^2 + v^2 + w^2$$

WH--Grubbs' approximation, MC--Monte Carlo (200 shots),  
WH and MC results taken from [8], [9]. P computed by ELLCOV.

TABLE 1 (Cont)

		R = 1, K = L = 0			
		$\sigma^2 = 1/2$	$\sigma^2 = 1$	$\sigma^2 = 2$	$\sigma^2 = 3$
H = 0	WH	.878	$u = v = 2w$ .618	.350	.236
	P	.878302	.625938	.351727	.229160
	MC	.890	.635	.365	.275
H = .5	WH	.738	.524	.314	.219
	P	.735871	.531421	.315323	.211615
	MC	.735	.520	.395	.270
H = 1	WH	.378	.308	.221	.171
	P	.378261	.319040	.226834	.166578
	MC	.380	.355	.265	.210
H = 2	WH	.005	.021	.044	.055
	P	9.5532(-3)	3.40282(-2)	5.94781(-2)	6.36591(-2)
	MC	.015	.040	.050	.070
H = 3	WH	.000	.000	.002	.006
	P	4.86(-6)	5.545(-4)	6.0278(-3)	1.26241(-2)
	MC	.000	.000	.000	.010
H = 0	WH	.740	$u = 2v = 3w$ .640	.415	.318
	P	.869274	.659413	.407256	.279152
	MC	.774	.662	.412	.275
H = .5	WH	.739	.568	.390	.299
	P	.747340	.586033	.378893	.265242
	MC	.750	.584	.366	.266
H = 1	WH	.447	.392	.312	.259
	P	.452305	.409374	.305022	.227522
	MC	.452	.408	.306	.223
H = 2	WH	.025	.068	.109	.123
	P	3.96655(-2)	9.20037(-2)	.127553	.123078
	MC	.044	.087	.122	.120
H = 3	WH	.000	.002	.015	.030
	P	3.428(-4)	6.6584(-3)	2.94238(-2)	4.40764(-2)
	MC	.001	.009	.029	.046

$$\sigma^2 = u^2 + v^2 + w^2$$

WH--Grubbs' approximation, MC--Monte Carlo (200 shots),  
 Last MC Group ( $u = 2v = 3w$ ) 1000 shots,  
 WH and MC results taken from [8], [9]. P computed by ELLCOV.

## III. COMPUTATION OF R (The inverse problem)

In this section we describe a procedure for finding R given P, H, K, L, u, v, w. For fixed H, K, L, u, v, w the radius R is a monotone increasing function of P and thus unique for a given P. The procedure for finding R requires the computation of P, as described in the previous section. Since that basically involves a time-consuming numerical triple integration a large effort is made to obtain good early approximations for R. Once such approximations are found uniform stepping, halving, Regula-falsi, and King's root finding procedure<sup>11</sup> are used to refine the estimates for R. The objective is to obtain R correctly to approximately 6-decimal digits.

In the discourse some details are omitted. They may be obtained from a detailed study of INVELLCOV, the BASIC program for computing R. A listing is given in Appendix C and a flowchart is shown in Figure 2 on page 18.

The unknown value of R will always be contained in a known interval. Initially crude lower and upper bounds,  $R_{\min}$  and  $R_{\max}$ , are found for R.

Let  $I \equiv \max[3\sqrt{\pi/2} Puvw, (\sqrt{\pi/2} PM)^3]$ ,  $M = \max(u, v, w)$ . Then

$$R \geq R_{\min} = \begin{cases} I^{1/3} & \text{if } P \leq 1/2 \\ I^{1/3} & \text{if } P > 1/2 \text{ and } I > D^3 = (H^2 + K^2 + L^2)^{3/2} \\ D & \text{if } P > 1/2 \text{ and } I \leq D^3. \end{cases} \quad (34)$$

The derivation of Equation 34 is similar to the methods used for deriving Tests #1 and #2 of Section II. See Appendix B.

For  $R_{\max}$ , two arrays  $A6(j)$ ,  $B6(j)$ ,  $j = 1, 2, \dots, 10$  are used.

$A6(1) = 5(-6)$	$B6(1) = .0266$
$A6(2) = 1(-4)$	$B6(2) = .0723$
$A6(3) = .01$	$B6(3) = .339$
$A6(4) = .1$	$B6(4) = .765$
$A6(5) = .3$	$B6(5) = 1.1933$
$A6(6) = .6$	$B6(6) = 1.717$
$A6(7) = .9$	$B6(7) = 2.5005$
$A6(8) = .999$	$B6(8) = 4.0335$
$A6(9) = .999999$	$B6(9) = 5.538$
$A6(10) = 1$	$B6(10) = 10$

The element  $B6(j)M$  gives the radius of a sphere centered at  $(0,0,0)$  for which  $A6(j) = P(B6(j)M, 0, 0, 0, M, M, M)$ . Then

$$R \leq R_{\max} = D + B6(J)M, \quad M = \max(u, v, w), \quad (35)$$

where  $J$  is the minimum integer  $j$  for which  $P \leq A6(j)$ . The plausibility of Equation 35 is easily seen from its 2-dimensional analog. In Figure 1, the inner ellipse contains  $P$  of the distribution. The outer ellipse with semi-axes  $qu$ ,  $qv$  contains  $A6(J)$  of the distribution. Then, from the figure, one easily concludes  $R < \sqrt{H^2 + K^2} + qs$ , where  $s = \max(u, v)$  and  $q$  corresponds to  $B6(J)$ .

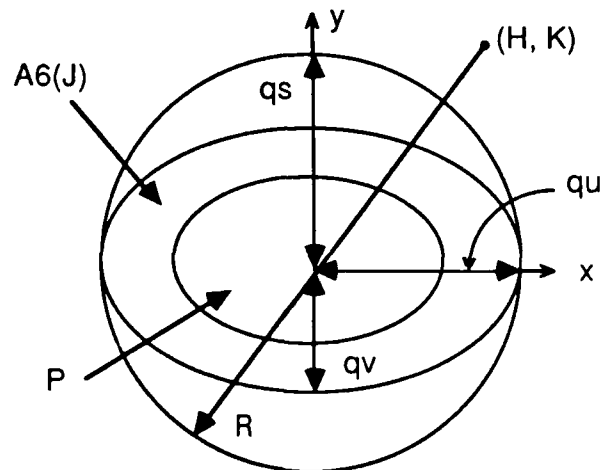


FIGURE 1: USED WITH (35)

$R_{\min}$  and  $R_{\max}$  establish crude bounds for  $R$ . An improvement is generally obtained by using an estimate for  $R$ , call it  $R_g$ , given by F.E. Grubbs.<sup>8,9</sup> His estimate depends on a percentage point of the chi-squared distribution which is available through the subprogram GAMINV. This subprogram is described in Reference 7. The quantity actually given by Grubbs is  $R_g^2$  which estimates  $R^2$  and is given by

$$R_g^2 = W_4 \{ [x(4/V_5, P) - 4/V_5] W_2 + N \}, \quad (36)$$

where

$$W_4 = u^2 + v^2 + w^2 \quad (37.1)$$

$$N = 1 + D^2/W_4 \quad (37.2)$$

$$V_4 = 2 \left\{ \frac{u^4}{W_4^2} \left[ 1 + 2 \left( \frac{H}{u} \right)^2 \right] + \frac{v^4}{W_4^2} \left[ 1 + 2 \left( \frac{K}{v} \right)^2 \right] + \frac{w^4}{W_4^2} \left[ 1 + 2 \left( \frac{L}{w} \right)^2 \right] \right\} \quad (37.3)$$

$$T_5 = 8 \left\{ \frac{u^6}{W_4^3} \left[ 1 + 3 \left( \frac{H}{u} \right)^2 \right] + \frac{v^6}{W_4^3} \left[ 1 + 3 \left( \frac{K}{v} \right)^2 \right] + \frac{w^6}{W_4^3} \left[ 1 + 3 \left( \frac{L}{w} \right)^2 \right] \right\} \quad (37.4)$$

$$W_2 = T_5/(2V_4) \quad (37.5)$$

$$V_5 = T_5^2/V_4^3, \quad (37.6)$$

and  $x(A, P)$  satisfies

$$P = \frac{1}{\Gamma(A)} \int_0^{x(A, P)} \exp(-t) t^{A-1} dt, \quad A = 4/V_5. \quad [1; p. 260] \quad (38)$$

If

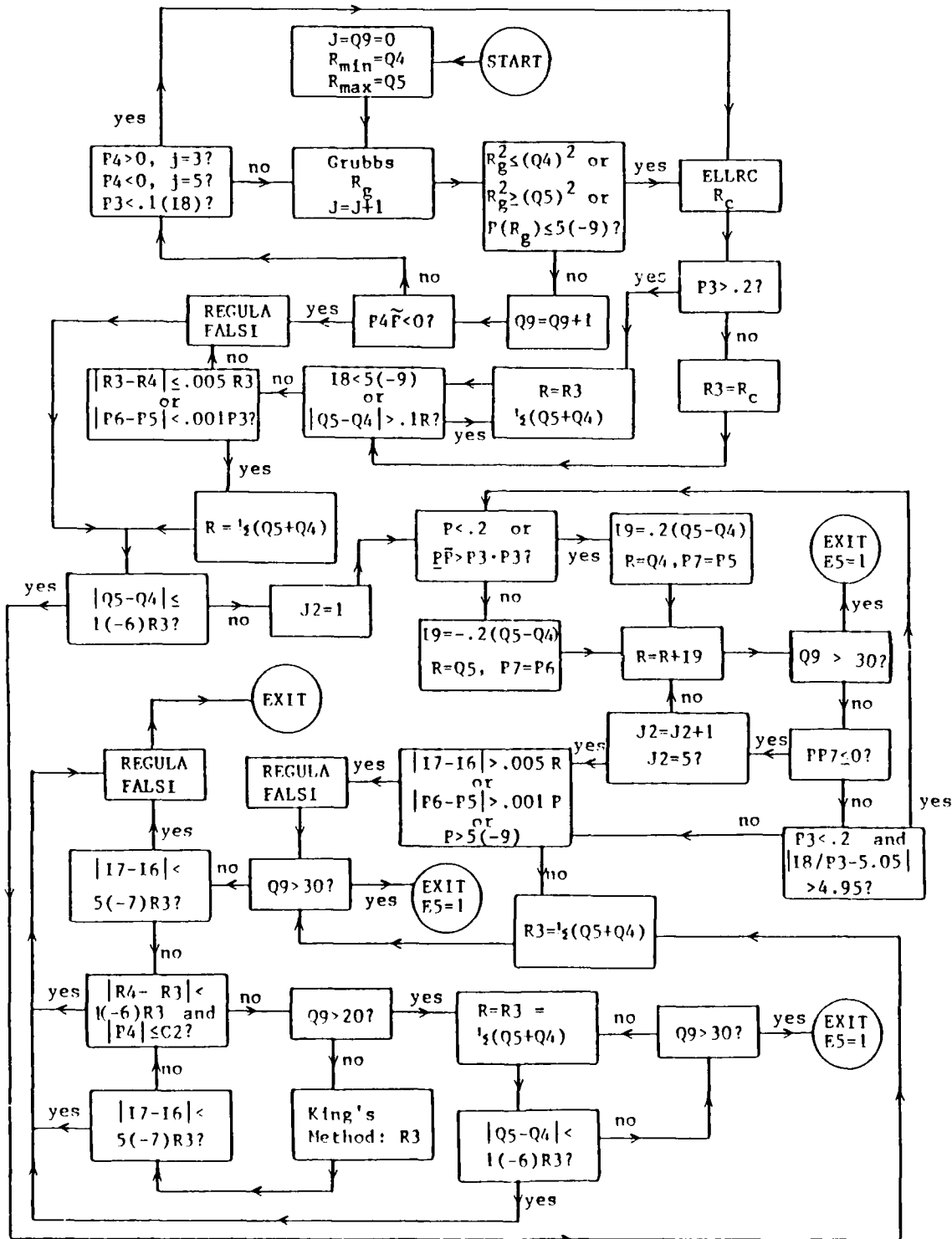
$$R_{\min}^2 < R_g^2 < R_{\max}^2, \quad (39)$$

then  $R_g$  is accepted as an improved estimate for  $R$ . If Equation 39 holds and  $P(R_g) < P$ , then  $R_g$  is stored in  $Q_4$  and  $R_{\max}$  is stored in  $Q_5$ ; if  $P(R_g) > P$ , then  $R_g$  is stored in  $Q_5$  and  $R_{\min}$  is stored in  $Q_4$ . Generally  $Q_4$  and  $Q_5$  bound  $R$  with  $Q_4 \leq R \leq Q_5$  (When King's procedure is used  $Q_4$  and  $Q_5$  still bound  $R$ , but their roles as lower and upper bounds may be reversed.). Refinements to  $Q_4$  and  $Q_5$  are obtained by using the procedures mentioned above. Some of the details are shown in the flow chart of Figure 2. A flow chart for King's root finding method is given in Figure 3 on page 19.

## NOTATION FOR FLOWCHART OF FIGURE 2 AND BASIC LISTING

$R_t$  denotes true value of  $R$  for a given value  $P = P_3$ .  $P_3 = P(R_t)$

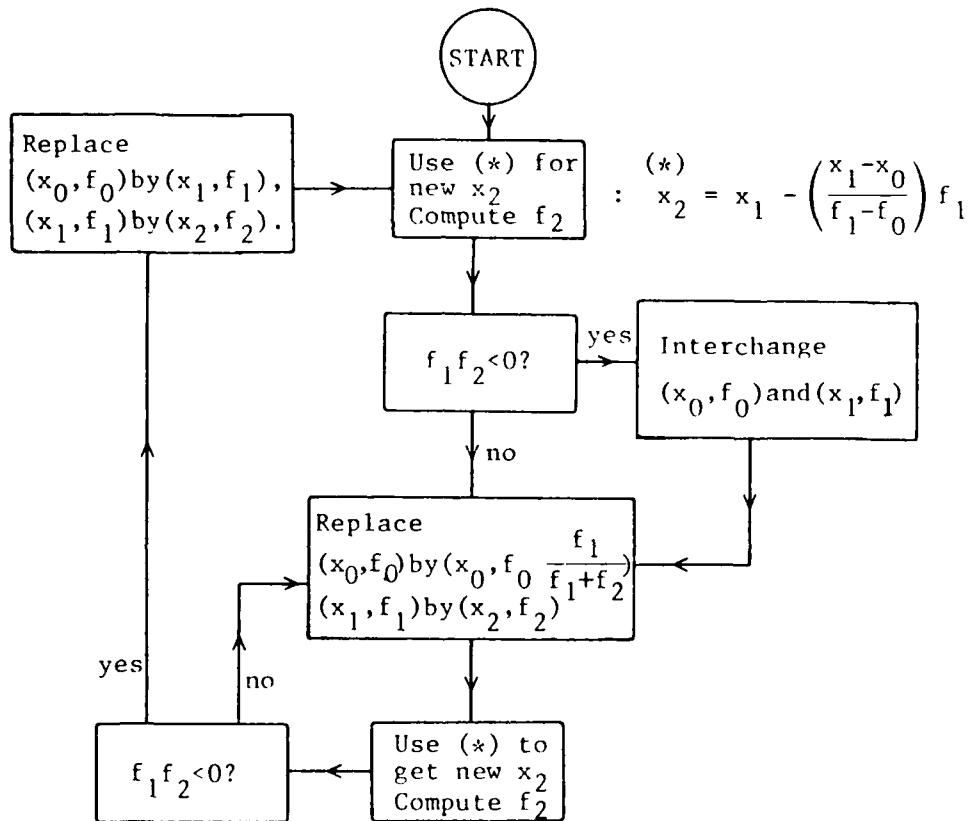
- C2 = 10 X8 (1 - P3),  $P_3 < .5$ ; C2 = 10 X8 P3,  $P_3 \geq .5$
- E4 = 0,  $1(-6) \leq P_3 \leq .999999$ ; E4 = 1,  $1(-6) > P_3 > .999999$
- E5 = 0, Less than 30 iterations to find  $R_3$ ; E5 = 1, 30 iterations are used to find best estimate for  $R_t$ . Final estimate  $R_3$  may not be accurate to 6-decimal digits.
- Hx - H of text; Hy - K of text; Hz - L of text.
- I6 -  $R_3$  with  $P_4 < 0$
- I7 -  $R_3$  with  $P_4 > 0$
- I8 -  $P(R_3)$ , latest value of  $P$
- I9 - Increment used in stepping procedure.  $I_9 = \pm .2 (Q_5 - Q_4)$ .
- K0 =  $H^2 + K^2 + L^2$ ;  $K_2 = \sqrt{K_0}$
- P - Previous value of I8 -  $P_3$
- P3 - Input value of  $P$
- P4 -  $P(R_3)$  or  $P(R_3) - P_3$
- P5 -  $P(Q_4) - P_3$
- P6 -  $P(Q_5) - P_3$
- P7 - P5 or P6
- $\bar{P}$  - Latest value of  $P > P_3$
- $\underline{P}$  - Latest value of  $P \leq P_3$
- Q4 - See Section III. Latest estimate for  $R_t$  such that  $P(Q_4) - P_3 < 0$ .
- Q5 - See Section III. Latest estimate for  $R_t$  such that  $P(Q_5) - P_3 > 0$ .
- Q9 - Contains number of iterations used to estimate  $R_t$ .
- R - Latest estimate for  $R_t$
- Rg - See Section III. Estimate from Grubb's approximation for  $R_t$ .
- Rc - See Section III. Lower estimate for  $R_t$  from ELLRC.
- R3 - The latest estimate for  $R_t$ .
- R4 - The next to latest estimate for  $R_t$ .
- $R_{min}$  - Starting lower bound for  $R_t$ .  $R_{max}$  - Starting upper bound for  $R_t^2$ .
- Sx - u of text
- Sy - v of text
- Sz - w of text
- M -  $\max(u, v, w)$
- T6 - Number of iterations used to obtain  $R_c$  using ELLRC.
- X8 - See Section III. Used to exit INVELLCOV if  $|P_4(R_3) - P_3| \leq X_8$ .



Not Shown:  $P(R)$  computed by ELLCOV. If  $|P(R) - P3| \leq X8$  then  $R$  is an acceptable estimate for  $R_t$ . INVELLCOV exited.

FIGURE 2. FLOW CHART FOR INVELLCOV

Input:  $x_0, f_0, x_1, f_1, (f_0 f_1 < 0)$



$x_2 \equiv$  latest estimate for  $\bar{x}$ ,  $f_n \equiv f(x_n)$ ,  $f(\bar{x}) = 0$

FIGURE 3. KING'S ROOT FINDING METHOD FOR  $f(x)$



In most cases  $R_g$  gives a very good estimate for  $R$ . In fact,  $R_g$  in many cases can be accepted as the final estimate for  $R$ , with no iterations required. If however Inequality 39 does not hold, (the right hand side of Equation 36 may even be negative) then we proceed as follows.

The constant  $R_p$  is determined so that

$$P = \int_{L-R_p}^{L+R_p} \int_{K-R_p}^{K+R_p} \int_{H-R_p}^{H+R_p} F(x,y,z,u,v,w) dx dy dz. \quad (40)$$

The subroutine ELLRC, using the Newton-Raphson method, is used to compute  $R_p$ . The right hand side of Equation 40 yields the probability  $P$  over the smallest cube, with center at  $(H,K,L)$  and sides parallel to the  $x$ ,  $y$ , and  $z$  axes, which contains the sphere of radius  $R_p$ . Hence  $R_p < R < \sqrt{3} R_p$ ; if  $R_p > R_{\min}$ , then  $Q4 = R_p$ , and if  $\sqrt{3} R_p < R_{\max}$  then  $Q5 = \sqrt{3} R_p$ .

In ELLRC the maximum number of iterations for  $R_p$  is set at 40. A maximum of 37 iterations has been observed.

Again, once  $R_p$  is computed, refinements are made by using halving, stepping, Regula-falsi and King's method to obtain a final estimate for  $R$ .

The iterations for  $R$  are generally terminated when  $Q4$  and  $Q5$  agree to at least 6 significant digits or if the latest estimate for  $R$ , say  $R_3$ , yields  $P(R_3)$  such that  $|P - P(R_3)| \leq X_8$ , where  $X_8$  is a small number which depends on  $P$ . For example, if  $P \leq 1(-5)$  then  $X_8 = 5(-10)$ , whereas if  $.001 < P < .999995$  then  $X_8 = 5(-7)$ . When exit conditions, some of which have just been described, are satisfied, then one more application of Regula-falsi is carried out before exiting INVELLCOV. See Figure 2 for more details, page 18.

The maximum number of iterations for  $R$  in INVELLCOV is set at 30. Extensive checking has never shown more than 20 iterations for any case. The average number of iterations is between 4 and 5. When  $P$  is very small ( $<.001$ ) more than the average number of iterations is generally needed.

INVELLCOV can be used with any reasonable input values of  $H$ ,  $K$ ,  $L$ ,  $u$ ,  $v$ ,  $w$ . The input variable  $P$  should satisfy  $|P - 1/2| \leq .499999$ . If  $P$  is in  $(0,1)$  and does not satisfy the above inequality, INVELLCOV may still give a result for  $R$ , but there is no assurance near 6 digit accuracy will be obtained.

Table 2 below contains some numerical results from INVELLCOV. It indicates the wide range of applicability of INVELLCOV.

TABLE 2

Item	u/v	v/w	R/w	H/u	K/v	L/w	P
1	1/2	2/3	3.183274/6	5/2	5/2	10/3	5(-6)
2	1/2	2/3	10.28906/6	5/2	5/2	10/3	5(-3)
3	1/2	2/3	16.60908/6	5/2	5/2	10/3	.1
4	1/2	2/3	23.39931/6	5/2	5/2	10/3	.5
5	1/2	2/3	30.48622/6	5/2	5/2	10/3	.9
6	1/2	2/3	40.77124/6	5/2	5/2	10/3	.999
7	1/2	2/3	48.44240/6	5/2	5/2	10/3	.999995
8	1(-3)	.05	1.171111	4000	20	5	5(-6)
9	1(-3)	.05	2.329994	4000	20	5	5(-3)
10	1(-3)	.05	3.855994	4000	20	5	.1
11	1(-3)	.05	5.103200	4000	20	5	.5
12	1(-3)	.05	6.364060	4000	20	5	.9
13	1(-3)	.05	8.155921	4000	20	5	.999
14	1(-3)	.05	9.472573	4000	20	5	.999995
15	2/5	5	6.184626	0	1	10	5(-6)
16	2/5	5	8.185500	0	1	10	5(-3)
17	2/5	5	9.733491	0	1	10	.1
18	2/5	5	11.72659	0	1	10	.5
19	2/5	5	15.45659	0	1	10	.9
20	2/5	5	22.95502	0	1	10	.999
21	2/5	5	29.02551	0	1	10	.999995
22	1(-5)	2(9)	2.1376073(4)	0	1(-4)	0	5(-6)
23	1(-5)	2(9)	1.2533239(7)	0	1(-4)	0	5(-3)
24	1(-5)	2(9)	2.5132270(8)	0	1(-4)	0	.1
25	1(-5)	2(9)	1.3489795(9)	0	1(-4)	0	.5
26	1(-5)	2(9)	3.2897073(9)	0	1(-4)	0	.9
27	1(-5)	2(9)	6.5810535(9)	0	1(-4)	0	.999
28	1(-5)	2(9)	9.1295755(9)	0	1(-4)	0	.999995

Values of R as computed by INVLLCOV.

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APPENDIX A  
DERIVATION OF (10 - 11) and (13 - 17)

Derivation of Equations 10, 11 and 13-17

Equations 10 and 11 refer to Case A in Section II. We have

$$\boxed{u = v = w} \quad D \equiv \sqrt{H^2 + K^2 + L^2}, \quad (\text{A-1})$$

and

$$P = \int_S F(x, y, z, u, v, w) \, dx \, dy \, dz, \quad (\text{A-2})$$

where the integration is carried out over the interior of the sphere

$$S : (x - H)^2 + (y - K)^2 + (z - L)^2 = R^2. \quad (\text{A-3})$$

Let

$$x = H + \tau \cos \theta \sin \phi$$

$$y = K + \tau \sin \theta \sin \phi$$

$$z = L + \tau \cos \phi,$$

$$0 \leq \tau \leq R, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi.$$

Then Equation A-2 becomes

$$P = \frac{1}{2\pi \sqrt{2\pi} uvw} \int_0^R \int_{\phi=0}^{\pi} \int_{\theta=0}^{2\pi} E(D/u) \exp\left\{-\frac{\tau}{u^2}[H \cos \theta \cos \phi + K \sin \theta \sin \phi + L \cos \phi]\right\} E(\tau/u) \tau^2 \sin \phi \, d\theta \, d\phi \, d\tau.$$

Taking advantage of spherical symmetry, we take  $H = K = 0$ ,  $L = D$ . Hence

$$P = \frac{1}{\sqrt{2\pi} u^3} \int_0^R \int_0^{\pi} E(D/u) \exp(-D\tau \cos \phi / u^2) E(\tau/u) \sin \phi \, d\phi \, \tau^2 \, d\tau. \quad (\text{A-4})$$

With  $\cos \phi = s$ ,

$$P = \frac{1}{\sqrt{2\pi} u^3} \int_0^R E(D/u) E(\tau/u) \left[ \int_{-1}^1 \exp(-D\tau s / u^2) \, ds \right] \tau^2 \, d\tau,$$

where

$$\int_{-1}^1 \exp(-D\tau s / u^2) \, ds = \frac{u^2}{D\tau} [\exp(D\tau / u^2) - \exp(-D\tau / u^2)].$$

Thus

$$P = \frac{1}{\sqrt{2\pi} Du} \int_0^R \{E[(D - \tau)/u] - E[(D + \tau)/u]\} \tau d\tau,$$

and

$$P = \frac{1}{2} \left\{ \operatorname{erf}\left(\frac{D+R}{\sqrt{2}u}\right) - \operatorname{erf}\left(\frac{D-R}{\sqrt{2}u}\right) - \sqrt{\frac{2}{\pi}} \frac{u}{D} \left[ E\left(\frac{D-R}{u}\right) - E\left(\frac{D+R}{u}\right) \right] \right\}, \quad D \neq 0. \quad (A-5)$$

If  $D = 0$ , then

$$P = \operatorname{erf}\left(\frac{R}{\sqrt{2}u}\right) - \sqrt{\frac{2}{\pi}} \frac{R}{u} E(R/u), \quad D = 0. \quad (A-6)$$

Now let

$$Y = \frac{E\left(\frac{D-R}{u}\right) - E\left(\frac{D+R}{u}\right)}{2DR/u^2} = E\left(\frac{D-R}{u}\right) \left\{ \frac{1 - \exp\left(\frac{-2DR}{u^2}\right)}{2DR/u^2} \right\}$$

Using the result in Equation A-5 gives Equation 10; then Equation A-6 or Equation 11 follows since  $Y = E(R/u)$  when  $D = 0$ .

#### Derivation of Equations 13-17

We have

$$\boxed{u = v \text{ and } H = K = 0} \quad u \neq w. \quad (A-7)$$

It is easily shown that

$$P_c(\bar{r}, 0, 0, u, u) = 1 - E(\bar{r}/u). \quad [5; \text{Eq. 2}] \quad (A-8)$$

Therefore with Relations A-7 and A-8 we have from Equation 6

$$P = \frac{1}{\sqrt{2\pi} w} \int_{L-R}^{L+R} E(z/w) [1 - E(\bar{r}/u)] dz,$$

where

$$\bar{r}^2 = R^2 - (L - z)^2 \geq 0.$$

Therefore

$$P = \frac{1}{2} [\operatorname{erf} L_4 - \operatorname{erf} L_5] - T, \quad L_4 = \frac{L+R}{\sqrt{2} w}, \quad L_5 = \frac{L-R}{\sqrt{2} w} \quad (\text{A-9})$$

$$T \equiv \frac{E(\sqrt{R^2 - L^2/u})}{\sqrt{2\pi} w} \int_{L-R}^{L+R} E(z/w) \exp\left[-\frac{1}{2}(2Lz - z^2)/u^2\right] dz. \quad (\text{A-10})$$

Two cases are now treated separately, namely  $u > w$  and  $u < w$ .

Let  $u > w$ . Then Equation A-10 becomes, after straight-forward algebra,

$$T = \frac{1}{\sqrt{2\pi} w} \exp\left\{-\frac{1}{2}\left[\left(\frac{R}{u}\right)^2 - \frac{L^2}{u^2 - w^2}\right]\right\} \frac{\sqrt{2}}{\alpha} \int_{S_0}^{F_1} \exp(-t^2) dt, \quad (\text{A-11})$$

$$F_1 = [\alpha(L+R) + \beta]/\sqrt{2}$$

$$S_0 = [\alpha(L-R) + \beta]/\sqrt{2}$$

where

$$\alpha = L_2/w, \quad \beta = \frac{Lw}{u^2 L_2}, \quad L_2 = \sqrt{|1 - (w/u)^2|}.$$

We also have

$$F_1 \equiv [\alpha(L+R) + \beta]/\sqrt{2} = \frac{(L+R)L_2}{\sqrt{2} w} + \frac{Lw}{\sqrt{2} L_2 u^2} = \frac{1}{\sqrt{2} w} (L/L_2 + RL_2) \quad (\text{A-12.1})$$

$$S_0 \equiv [\alpha(L-R) + \beta]/\sqrt{2} = \frac{1}{\sqrt{2} w} (L/L_2 - RL_2), \quad (\text{A-12.2})$$

with  $L_2$  given by Equation 19.1. Then from Equation A-9, using Equation A-11 and Equation A-12, we get

$$P = \frac{1}{2} \left\{ \operatorname{erf} L_4 - \operatorname{erf} L_5 - \frac{1}{L_2} E(R/u) E[L/(uL_2)] (\operatorname{erf} F_1 - \operatorname{erf} S_0) \right\} \quad (\text{A-13})$$

which is Equation 13.

Another form for Equation A-13, namely Equation 14, is obtained by replacing  $\operatorname{erf} F_1 - \operatorname{erf} S_0$  by  $\operatorname{erfc} S_0 - \operatorname{erfc} F_1$ , and then multiplying  $\operatorname{erfc} S_0$  by  $\exp(S_0^2) \exp(-S_0^2)$ , and  $\operatorname{erfc} F_1$  by  $\exp(F_1^2) \exp(-F_1^2)$ . Combining the results with the remaining exponentials in Equation A-13 gives Equation 14, where use is made of the relation

$$\frac{1}{2} \left\{ (R/u)^2 - L^2/(u^2 - w^2) + [\alpha(L \pm R) + \beta]^2 \right\} = \frac{1}{2} [(L \pm R)/w]^2.$$

When  $L = 0$ , Equation A-13 reduces to

$$P = \operatorname{erf}(R/\sqrt{2} w) - (1/L_2) E(R/u) \operatorname{erf}(RL_2/\sqrt{2} w). \quad (\text{A-14})$$

Now consider  $w > u$ . Starting with Equation A-10, we have

$$T = E(R/u) E(L/\sqrt{w^2 - u^2}) \frac{1}{\sqrt{2\pi} w} \int_{L-R}^{L+R} \exp \left[ \frac{1}{2} (\alpha z - \beta)^2 \right] dz. \quad (\text{A-15})$$

Note here that the argument of the exponential in the integrand is positive. Proceeding as above,  $T$  is obtained in terms of Dawson's integral denoted here by  $\operatorname{daws}(x)$ .

$$T = \frac{1}{\sqrt{\pi} L_2} E(R/u) E \left[ L / \sqrt{w^2 - u^2} \right] \left[ \exp(F_1^2) \operatorname{daws}(F_1) - \exp(S_0^2) \operatorname{daws}(S_0) \right],$$

$$\operatorname{daws}(x) \equiv \exp(-x^2) \int_0^x \exp(t^2) dt, \quad \operatorname{daws}(-x) = -\operatorname{daws}(x), \quad [1; \text{p. 298}], [3].$$

We also have using Equation A-12

$$S_0 = \frac{-1}{\sqrt{2} uw \sqrt{|u^2 - w^2|}} [R(w^2 - u^2) - Lu^2]$$

$$F_1 = \frac{1}{\sqrt{2} uw \sqrt{|u^2 - w^2|}} [R(w^2 - u^2) + Lu^2].$$

Thus using the relation

$$\begin{aligned} \frac{1}{2} \left\{ \left( \frac{R}{u} \right)^2 + \frac{L^2}{(w^2 - u^2)} - \frac{1}{u^2 w^2 (w^2 - u^2)} [R^2 (w^2 - u^2)^2 \pm 2RLu^2 (w^2 - u^2) + u^4 L^2] \right\} \\ = \frac{1}{2} \left\{ \left( \frac{R}{u} \right)^2 \left( 1 - \frac{(w^2 - u^2)}{w^2} \right) \pm \frac{2RL}{w^2} + \frac{L^2}{(w^2 - u^2)} \left( 1 - \frac{u^2}{w^2} \right) \right\} = \left[ \frac{L \pm R}{\sqrt{2} w} \right]^2 \end{aligned}$$

Equation A-15 reduces to

$$T = \frac{1}{2L_2} \frac{2}{\sqrt{\pi}} E \left( \frac{L - R}{w} \right) [\operatorname{daws}(F_1) - \exp(-2RL/w^2) \operatorname{daws}(S_0)], \quad (\text{A-16})$$

which agrees with the corresponding term in Equation 16.



For  $L = 0$ , with  $w > u$ , it is easy to conclude from Equation A-9 and Equation A-16 that

$$P = \operatorname{erf} L_4 - \frac{2}{\sqrt{\pi}} \frac{1}{L_2} E(R/w) \operatorname{daws}(RL_2/\sqrt{2}w), \quad (\text{A-17})$$

which is Equation 17.

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APPENDIX B  
DISCUSSION OF TESTS #1, #2, #3, #4

Discussion of Tests #1, #2, #3, #4

Test #1: From Equation 4 with  $E(x/u) E(y/v) E(z/w) < 1$ , we have

$$P < \frac{4}{3} \pi R^3 / (2\pi \sqrt{2\pi} u v w) \leq Z_4, \quad Z_4 = 2.5(-8),$$

or

$$R^3 \leq 1.5 \sqrt{2\pi} uvw Z_4 \quad (\Rightarrow P < Z_4). \quad (B-1)$$

The relations in Inequality B-1 lead to Test #1.

Also, from Equation 6, setting  $E(z/w) P_c = 1$  with  $w = \max(u, v, w)$

$$P < \frac{1}{\sqrt{2\pi} M} [(L + R) - (L - R)] = \sqrt{\frac{2}{\pi}} \frac{R}{M} \leq Z_4, \quad M = \max(u, v, w),$$

$$R < \sqrt{\frac{\pi}{2}} Z_4 M. \quad (B-2)$$

The last inequality and Inequality B-1 are used in Equation 34.

Test #2: Note if  $D > R$  then  $P < 1/2$  and the sphere  $S$  does not contain the origin. Therefore, we have for  $D > R$

$$E(x/u) E(y/v) E(z/w) \leq E(x/M) E(y/M) E(z/M) \leq E[(D - R)/M]$$

and it follows

$$P < \frac{1}{2\pi \sqrt{2\pi} uvw} \int_S E\left(\frac{D - R}{M}\right) dx dy dz = \frac{4\pi R^3}{6\pi \sqrt{2\pi} uvw} E\left(\frac{D - R}{M}\right) \leq Z_4. \quad (B-3)$$

Then Test #2 follows directly from Inequality B-3.

Test #3: This follows by recognizing that the sphere  $S$  is located outside the region which contains  $1 - Z_4$  of the distribution. It is shown by Equation 26 and Equation 27 that if  $a$  is determined by  $\text{erfc}(a/\sqrt{2}) = \epsilon$  then the entire region  $\bar{U}$  above the plane  $z = a w$  contains  $\epsilon/2$  of the distribution. Since  $z = L - R > 0$  locates the point of the sphere closest to the  $xy$ -plane, and if  $L - R > \sqrt{2} w F_2(\sqrt{2} F_2 = a)$ , then the sphere  $S$  must be entirely contained in the region  $\bar{U}$ . Here  $L$  has been used for  $h$ , and  $w$  has been used for  $T$ .

Test #4: This test is based on the fact that if  $P \approx 1$  and  $u, v$  or  $w$  increases, then for fixed  $R$ ,  $P$  must decrease. Thus if we replace  $u, v$ , and  $w$  by  $M$  to obtain  $\bar{P}$ , then  $\bar{P} \leq P$ . If  $\bar{P} \geq 1 - Z_4$ , then  $P \geq 1 - Z_4$ .

A heuristic argument follows that indicates if

$$R^2 > H^2 + K^2 + L^2, \quad (B-4)$$

then

$$P = P(R, H, K, L, u, v, w) \geq P(R, H, K, L, M, M, M), \quad M = \max(u, v, w) \quad (B-5)$$

Without loss of generality, assume  $w < M$  and let

$$P = P_0 \equiv P(R_0, H, K, 0, u, v, w). \quad (B-6)$$

Then it is easy to show from Equation 8 (with  $L = 0$ ) that

$$\frac{\partial P_0}{\partial w} = \int_{-R/\sqrt{2}w}^{R/\sqrt{2}w} \frac{\partial P_c}{\partial \xi} \frac{\partial \xi}{\partial w} dt < 0, \quad (B-7)$$

where

$$\xi = \sqrt{R^2 - 2w^2 t^2} \quad \text{and} \quad \frac{\partial P_c(\xi, H, K, u, v)}{\partial \xi} > 0 \quad (B-8)$$

Thus from Equation B-7,

$$P = P_0(R_0, H, K, 0, u, v, w) > P(R_0, H, K, 0, u, v, M). \quad (B-9)$$

Then by continuity, we have for sufficiently small  $\bar{\delta}$

$$P = P(\bar{R}, H, K, \bar{\delta}, u, v, w) > P(\bar{R}, H, K, \bar{\delta}, u, v, M). \quad (B-10)$$

Now consider a sequence of spheres  $\{S_i\}$  with centers  $(H, K, \delta_i)$  and radii  $R_i$ , such that  $\delta_i$  is an arbitrary increasing set of positive numbers with  $0 \leq \delta_i \leq L$ ,  $R_0 \leq R_i \leq R$  and

$$P = P(R_i, H, K, \delta_i, u, v, w). \quad (B-11)$$

Note that by Inequality B-4 all  $S_i$  contain the origin.

If  $P = P(R_i, H, K, \delta_i, u, v, w) > P(R_i, H, K, \delta_i, u, v, M)$  holds for all elements  $S_i$ , then a simple argument establishes Inequality B-5. Suppose however for some  $R_i = \hat{R}$ ,  $\delta_i = \hat{\delta}$  the last inequality does not hold. Since Inequality B-9 always holds, there must exist by continuity a sphere  $\hat{S}$  with  $\hat{R}$ ,  $\hat{\delta}$ , ( $\bar{R} < \hat{R} < \hat{R} \leq R$ ,  $\bar{\delta} < \hat{\delta} < \hat{\delta} \leq L$ ) such that

$$P = P(\hat{R}, H, K, \hat{\delta}, u, v, w) = P(\hat{R}, H, K, \hat{\delta}, u, v, M). \quad (B-12)$$

But Equation B-12 is impossible, since it implies that the two different cumulative normal distributions over the same sphere can give the same value of  $P$ . Indeed, denote the two different density functions of Equation (B-12) by  $I_w$  and  $I_M$ . These functions, with 3 independent variables and hence not easily visualized, intersect along a surface  $T$  bounded by a simple closed curve. From Equation B-12 and Inequality B-4 with the integrations of  $I_w$  and  $I_M$  carried out over the interior of  $\hat{S}$ , we have

$$P = J_w + A_w = J_M + A_M, \quad (B-13)$$

where  $J_w$  ( $J_M$ ) denotes the integral of  $I_w$  ( $I_M$ ) over that part of  $\hat{S}$  which contains the origin and is bounded by  $T$ ;  $A_w$  ( $A_M$ ) denotes the remaining contribution from the integration of  $I_w$  ( $I_M$ ) over that part of  $\hat{S}$  bounded by  $T$  and not containing the origin. For  $w < M$ , then  $I_w > I_M$  at the origin and

$$J_w > J_M, \text{ or } J_w = J_M + a \quad a \geq 0; \quad (B-14)$$

$$A_w < A_M, \text{ or } A_w = A_M - b \quad b \geq 0. \quad (B-15)$$

But  $a > b$ , since the integration of  $I_w$  and  $I_M$  over all xyz-space must equal one. This leads to a contradiction of Equation B-13.

This argument is easily seen in the one dimensional case. We use the same notation as above for the corresponding quantities in one dimension. See the figure below.

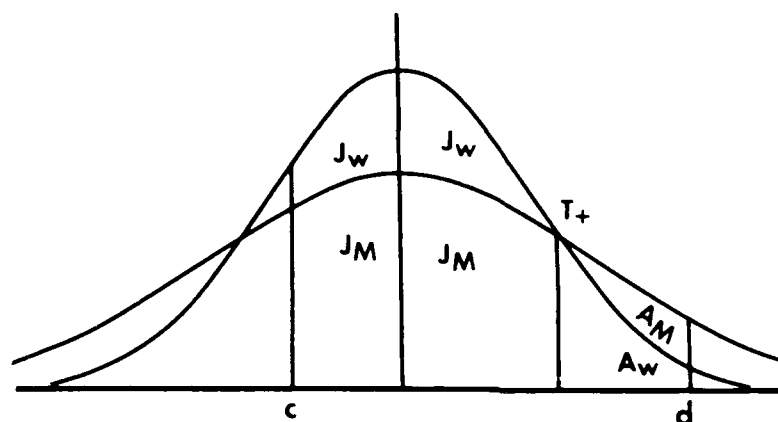


FIGURE 4: NORMAL CURVES WITH UNEQUAL STANDARD DEVIATIONS

Let

$$D(t/y) \equiv \frac{1}{\sqrt{2\pi} y} E(t/y) \quad (B-16)$$

$$J_w(c, T_+) = \int_c^{T_+} D(t/w) dt, \quad J_M(c, T_+) = \int_c^{T_+} D(t/M) dt$$

$$A_w = \int_{T_+}^d D(t/w) dt, \quad A_M = \int_{T_+}^d D(t/M) dt.$$

Clearly with  $c < 0$ ,  $d > 0$  (corresponding to  $R^2 > H^2 + K^2 + L^2$ ) we have

$$J_w > J_M, \quad A_w < A_M, \quad \text{and} \quad J_w(c, d) > J_M(c, d).$$

Thus by the contradiction above we conclude that

$$P(R, H, K, L, u, v, w) > P(R, H, K, L, u, v, M)$$

Then Inequality B-5 easily follows by carrying out the same argument using the  $\min(u, v)$ , say  $v$ , in place of  $w$  to show the last inequality below,

$$P(R, H, K, L, u, v, w) > P(R, H, K, L, M, v, M) > P(R, H, K, L, M, M, M).$$

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APPENDIX C  
H-P BASIC LISTINGS OF ELLCOV AND INVELLCOV

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1925 ! ARRAYS X(*), Y(*), A6(*), B(*) MUST BE CALLED WITH BASIC SUBROUTINE
1926 ! IN MAIN PROGRAM IN ORDER TO USE ELLCOV OR INVELLCOV.
1930 DIM S2(20), Hx(20), H(3), S(3), A5(3), B5(3), T5(3), P(7), W8(3)
1935 COM X(43), Y(43), A6(10), B6(10), Q9, V, A7, P4, Ve, T6, P1, T9
1940 ! GAUSSIAN CONSTANTS STORED IN "GAUSS". USED IN "SUBGquad".
1945 X(1)=.238619186083
1950 X(2)=.661209386466
1955 X(3)=.932469514203
1960 X(4)=.183434642496
1965 X(5)=.525532409916
1970 X(6)=.796666477414
1975 X(7)=.960289856498
1980 X(8)=.125233408511
1985 X(9)=.367831498998
1990 X(10)=.587317954287
1995 X(11)=.769902674194
2000 X(12)=.90411725637
2005 X(13)=.981560634247
2010 X(14)=9.50125098376E-2
2015 X(15)=.281603550779
2020 X(16)=.458016777657
2025 X(17)=.617876244403
2030 X(18)=.755404408355
2035 X(19)=.865631202388
2040 X(20)=.944575023073
2045 X(21)=.989400934992
2050 X(22)=7.65265211335E-2
2055 X(23)=.227785851142
2060 X(24)=.373706088715
2065 X(25)=.510867001951
2070 X(26)=.636053680727
2075 X(27)=.74633190646
2080 X(28)=.839116971822
2085 X(29)=.912234428251
2090 X(30)=.963971927278
2095 X(31)=.993128599185
2100 X(32)=6.40568928626E-2
2105 X(33)=.191118867474
2110 X(34)=.315042679696
2115 X(35)=.433793507626
2120 X(36)=.545421471389
2125 X(37)=.648093651937
2130 X(38)=.740124191579
2135 X(39)=.820001985974
2140 X(40)=.886415527004
2145 X(41)=.938274552003
2150 X(42)=.974728555971
2155 X(43)=.995187219997
2160 Y(1)=.467913934573
2165 Y(2)=.360761573048
2170 Y(3)=.171324492379
2175 Y(4)=.362683783378
2180 Y(5)=.313706645878
2185 Y(6)=.222381034453
2190 Y(7)=.10122853629
2195 Y(8)=.249147045813
2200 Y(9)=.233492536538

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2205 Y(10)=.203167426723  
2210 Y(11)=.160078328543  
2215 Y(12)=.106939325995  
2220 Y(13)=4.71753363865E-2  
2225 Y(14)=.189450610455  
2230 Y(15)=.182603415045  
2235 Y(16)=.169156519395  
2240 Y(17)=.149595988817  
2245 Y(18)=.124628971256  
2250 Y(19)=9.51585116825E-2  
2255 Y(20)=6.22535239386E-2  
2260 Y(21)=2.71524594118E-2  
2265 Y(22)=.152753387131  
2270 Y(23)=.149172986473  
2275 Y(24)=.142096109318  
2280 Y(25)=.131688638449  
2285 Y(26)=.118194531962  
2290 Y(27)=.101930119817  
2295 Y(28)=8.32767415767E-2  
2300 Y(29)=6.26720483341E-2  
2305 Y(30)=4.06014298004E-2  
2310 Y(31)=1.76140071392E-2  
2315 Y(32)=.127938195347  
2320 Y(33)=.125837456347  
2325 Y(34)=.121670472928  
2330 Y(35)=.115505668054  
2335 Y(36)=.107444270116  
2340 Y(37)=9.76186521041E-2  
2345 Y(38)=.086190161532  
2350 Y(39)=7.33464814111E-2  
2355 Y(40)=5.92985349154E-2  
2360 Y(41)=4.42774388174E-2  
2365 Y(42)=2.85313886289E-2  
2370 Y(43)=.0123412298  
2375 A6(1)=5E-6  
2380 A6(2)=.0001  
2385 A6(3)=.01  
2390 A6(4)=.1  
2395 A6(5)=.3  
2400 A6(6)=.6  
2405 A6(7)=.9  
2410 A6(8)=.999  
2415 A6(9)=.999999  
2420 A6(10)=1  
2425 B6(1)=.0266 !.0372  
2430 B6(2)=.0723 !.15  
2435 B6(3)=.339 !.474  
2440 B6(4)=.765 !.9  
2445 B6(5)=1.1933 !1.686  
2450 B6(6)=1.717  
2455 B6(7)=2.5005 !3.662  
2460 B6(8)=4.0335 !6.215  
2465 B6(9)=5.538 !8.84  
2470 B6(10)=10  
2475 RETURN

```

2480 ! *****START OF INVERSE PROCEDURE*****
2485 ! THIS PROGRAM IS CALLED INVELLCOV. (X,Y,Z) IS A POINT IN A CARTESIAN CO-
2486 ! ORDINATE SYSTEM. INVELLCOV RETURNS THE RADIUS OF THE SPHERE WITH CENTER
2490 ! (Hx,Hy,HZ) WHICH HAS P3 OF THE NORMAL ELLIPSOIDAL DISTRIBUTION WITH
2491 ! MEAN (0,0,0) AND STANDARD DEVIATIONS Sx,Sy,Sz IN THE X,Y,Z DIRECTIONS,
2495 ! RESPECTIVELY. THE RADIUS R IS GENERALLY GIVEN CORRECTLY TO AT LEAST 6-
2496 ! DECIMAL DIGITS. THE INPUT P3 SHOULD SATISFY  $ABS(P3-1/2) < .499999$ . IF
2500 ! P3 IS IN [0,1] BUT THE ABOVE INEQUALITY IS NOT SATISFIED, THEN THE
2501 ! OUTPUT R MAY NOT BE CORRECT TO 6-DECIMAL DIGITS. A MAXIMUM OF 30 ITERA-
2505 ! TIONS IS ALLOWED. IF E5=1 THEN 30 ITERATIONS HAVE OCCURRED. IF E4=1
2506 ! THEN P3 IS IN (0,1) BUT NOT IN  $[1E-6, .999999]$ . IF P3 IS NOT IN [0,1]
2510 ! THEN A VALUE OF -1E-99 IS RETURNED FOR R. SUBROUTINES USED ARE ELLCOV,
2511 ! ELLRC, AND GAMINV.
2515 DEF FNInvellcov(P3,Hx,Hy,HZ,Sx,Sy,Sz,E4,E5)
2520 COM X(43),Y(43),A6(10),B6(10),Q9,V,A7,P4,Ve,T6,P1,T9
2525 K0=Hx*Hx+Hy*Hy+HZ*HZ
2530 K2=SQR(K0)
2535 S=MAX(Sx,Sy,Sz)
2540 A4=.797884560803! SQR(2/PI)
2545 R3=E5=E4=0
2550 W4=Sx*Sx+Sy*Sy+Sz*Sz
2555 IF ABS(P3-.5)<=.499999 THEN 2595
2560 IF ABS(P3-.5)<=.5 THEN 2575
2565 R3=-1E99 !****"P3 IS UNACCEPTABLE"
2570 RETURN R3
2575 E4=1 !****P3 NOT IN  $[1E-6, .999999]$ .
2580 IF P3=0 THEN RETURN 0
2585 IF P3<1 THEN 2595
2590 RETURN 1E99
2595 Q9=0
2600 X8=.00000005
2605 IF P3>.00001 THEN 2620
2610 X8=5E-10
2615 GOTO 2650
2620 IF P3>.001 THEN 2635
2625 X8=.0000000025
2630 GOTO 2650
2635 IF P3<.999995 THEN 2650
2640 X8=.000000005
2645 IF P3>.9999975 THEN X8=1E-8 !NEW
2650 P5=P3*S/A4
2655 Q4=MAX(3*P3*Sx*Sy*Sz/A4,P5*P5*P5)
2660 P4=0
2665 IF P3>=.5 THEN 2685
2670 C2=10*X8*(1-P3)
2675 R3=Q4^(1/3)
2680 GOTO 2700
2685 R3=K2
2690 IF Q4>K2*K2*K2 THEN R3=Q4^(1/3)
2695 C2=10*X8*P3
2700 Rmin=Q4=R3
2705 P4=FNellcov(R3,Hx,Hy,HZ,Sx,Sy,Sz,K0,K2,S)
2710 IF P4>=P3 THEN RETURN R3
2715 P5=P4-P3
2720 FOR J=1 TO 10
2725 IF P3<=A6(J) THEN 2735

```

```

2730 NEXT J
2735 R3=K2+B6(J)*S
2740 Rmax=Q5=R3
2745 T6=0
2750 P4=FNE11cov(R3,Hx,Hy,Hz,Sx,Sy,Sz,K0,K2,S)
2755 IF P4<=P3 THEN RETURN R3
2760 P6=P4-P3
2765 I6=Q4
2770 I7=Q5
2775 ! *****GRUBB'S ESTIMATE FOR R3*****
2780 D2=0
2785 D3=1.1
2790 IF P3<=.8 THEN 2810
2795 D3=1.25
2800 IF P3<=.9 THEN 2810
2805 D3=1.9
2810 D4=P4-P3
2815 M0=1+K0/W4 ! 1+(Hx*Hx+Hy*Hy+Hz*Hz)/W4
2820 Wx=Sx*Sx/W4
2825 Wy=Sy*Sy/W4
2830 Wz=Sz*Sz/W4
2835 Gx=Hx/Sx
2840 Gy=Hy/Sy
2845 Gz=Hz/Sz
2850 V4=2*(Wx*Wx*(1+2*Gx*Gx)+Wy*Wy*(1+2*Gy*Gy)+Wz*Wz*(1+2*Gz*Gz))
2855 V5=8*(Wx*Wx*Wx*(1+3*Gx*Gx)+Wy*Wy*Wy*(1+3*Gy*Gy)+Wz*Wz*Wz*(1+3*Gz*Gz))
2860 W2=.5*V5/V4
2865 V5=V5*V5/(V4*V4*V4)
2870 R4=R3
2875 IF P4=1 THEN P4=.999999999
2880 R3=W4*((FNGaminv(4/V5,X,0,P4,1-P4,Ierr)-4/V5)*W2+M0)
2885 IF (R3<=Q4*Q4) OR (R3>=Q5*Q5) THEN 3090
2890 R3=SQR(R3)
2895 IF ABS(R3-R4)<5E-9*R3 THEN 3090
2900 P4=FNE11cov(R3,Hx,Hy,Hz,Sx,Sy,Sz,K0,K2,S)
2905 I8=P4
2910 IF P4>0 THEN Q9=Q9+1
2915 IF P4>5E-9 THEN 2940
2920 IF R3<=Q4 THEN 3090
2925 Q4=R3
2930 P5=P4-P3
2935 GOTO 3090
2940 W5=P4-P3
2945 IF ABS(W5)<=X8 THEN RETURN R3
2950 IF W5<0 THEN 3020
2955 Q5=R3
2960 I7=Q5
2965 P6=W5
2970 IF D2<0 THEN 3175
2975 IF Q9>2 THEN 3090
2980 IF I8>10*P3 THEN 3090
2985 IF (P3>.01) OR (P4=1) THEN 3005
2990 IF ABS(W5)>.1*P3 THEN 3005

```

```

2995     P4=D4=P3-2*W5
3000     GOTO 3010
3005     P4=D4=D4^D3
3010     D2=1
3015     GOTO 2870
3020     Q4=R3
3025     I6=Q4
3030     P5=W5
3035     D5=2-D3
3040     IF D2>0 THEN 3175
3045     IF (Q9>4) OR (I8<.01*P3) THEN 3090
3050     IF (P3>.01) OR (P4=0) THEN 3070
3055     IF ABS(W5)>.1*P3 THEN 3070
3060     P4=D4=P3-2*W5
3065     GOTO 3075
3070     P4=D4=D4^D5
3075     D2=-1
3080     GOTO 2870
3085     ! *****ESTIMATE FOR R3 USING CUBE*****
3090     P11=P1=FNE11nc(Hx,Hy,HZ,Sx,Sy,Sz,P3,0,Q5,T6)
3095     R4=R3
3100     IF T6<=Ve THEN 3110      ! USED FOR MAX. OF T6.
3105     Ve=T6                    ! USED FOR MAX. OF T6.
3110     IF P1<=Q4 THEN 3130
3115     R3=P1
3120     GOSUB 3740
3125     IF P4>=0 THEN RETURN P1
3130     P1=1.732051*P1
3135     IF P1<Q5 THEN Q5=P1
3140     IF (P3<=.2) AND (P11=Q4) THEN 3160
3145     R4=R3
3150     R3=.5*(Q4+Q5)
3155     GOSUB 3740      ! COMPUTES ELLCOV(R3) AND MAKES PROPER STORAGES.
3160     IF ABS(P4)<=X8 THEN 3660
3165     IF (I8<=5E-9) OR (ABS(Q5-Q4)>.1*R3) THEN 3145
3170     IF (ABS(R3-R4)<=.005*R3) OR (ABS(P6-P5)<=.001*P3) THEN 3190
3175     R4=R3
3180     R3=Q4-(Q4-Q5)/(P5-P6)*P5      ! REGULA-FALSI
3185     GOTO 3200
3190     R4=R3
3195     R3=.5*(Q4+Q5)
3200     GOSUB 3740
3205     IF ABS(P4)<=X8 THEN 3660
3210     IF ABS(Q5-Q4)<=.000001*R3 THEN 3315
3215     ! *****STEPPING PROCEDURE*****
3220     I9=.2*(Q5-Q4)
3225     P7=P5
3230     R3=Q4
3235     IF (P3<.2) OR (P6+P5>0) THEN 3255
3240     I9=-I9
3245     P7=P6
3250     R3=Q5

```

```

3255   FOR J2=1 TO 4
3260     R4=R3
3265     R3=R3+I9
3270     T9=3270
3275     GOSUB 3740
3280     IF ABS(P4)<=X8 THEN 3660
3285     IF Q9>30 THEN 3725
3290     IF P4*P7>0 THEN 3305
3295     IF (P3<.2) AND (ABS(I8/P3-5.05)>4.95) THEN 3220
3300     GOTO 3310
3305   NEXT J2
3310   IF (ABS(I7-I6)>.005*R3) OR (ABS(P6-P5)>.001*P3) OR (I8>5E-9) THEN 3335
3315   R3=.5*(Q4+Q5) ! R3=(2*Q4+Q5)/3
3320   GOTO 3345
3325   ! *****KING'S PROCEDURE*****
3330   R4=R3
3335   R3=Q5-P6*(Q5-Q4)/(P6-P5)
3340   T9=3400
3345   IF Q9>=30 THEN 3725
3350   P4=FNE11cov(R3,Hx,Hy,HZ,Sx,Sy,Sz,K0,K2,S)
3355   IF P4<>0 THEN Q9=Q9+1
3360   P4=P4-P3
3365   IF ABS(P4)<=X8 THEN 3660
3370   IF P4>0 THEN 3390
3375   I6=R3
3380   I3=-P4
3385   GOTO 3400
3390   I7=R3
3395   I4=P4
3400   IF I7<=I6 THEN 3555
3405   IF ABS(I7-I6)<=.0000005*R3 THEN 3660
3410   IF (ABS(R4-R3)<=.000001*R3) AND (ABS(P4)<=C2) THEN 3660
3415   IF Q9>20 THEN 3605
3420   IF P4*P6>0 THEN 3455
3425   A6=Q5
3430   Q5=Q4
3435   Q4=A6
3440   A6=P5
3445   P5=P6
3450   P6=A6
3455   P5=P5*P6/(P6+P4)
3460   P6=P4
3465   Q5=R3
3470   R4=R3
3475   R3=Q5-(Q4-Q5)*P6/(P5-P6)
3480   T9=3480
3485   IF Q9>=30 THEN 3725
3490   P4=FNE11cov(R3,Hx,Hy,HZ,Sx,Sy,Sz,K0,K2,S)
3495   IF P4<>0 THEN Q9=Q9+1
3500   P4=P4-P3
3505   IF (ABS(P4)<=X8) AND (P5<>-P3) AND (P6<>1-P3) THEN 3660
3510   IF P4>0 THEN 3530
3515   I6=R3
3520   I3=-P4
3525   GOTO 3540

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3530   I7=R3
3535   I4=P4
3540   IF ABS(I7-I6)<=.0000005*R3 THEN 3660
3545   IF Q9>20 THEN 3605
3550   IF I7>I6 THEN 3570
3555   R3=I6
3560   IF ABS(I4)<ABS(I3) THEN R3=I7
3565   RETURN R3
3570   IF (ABS(R4-R3)<=.000001*R3) AND (ABS(P4)<=C2) THEN 3660
3575   IF P4*P6>=0 THEN 3455
3580   Q4=Q5
3585   Q5=R3
3590   P5=P6
3595   P6=P4
3600   GOTO 3330
3605   Q4=I6
3610   Q5=I7
3615   R4=R3
3620   R3=.5*(Q4+Q5)
3625   T9=3685
3630   GOSUB 3740
3635   IF ABS(Q5-Q4)<=.000001*R3 THEN 3660
3640   IF ABS(P4)<=X8 THEN 3660
3645   IF Q9>30 THEN 3725
3650   GOTO 3615
3655   ! *****EXITS*****8
3660   IF P4*P6<0 THEN 3680
3665   IF P4*P5>0 THEN 3700
3670   R3=R3-(Q4-R3)*P4/(P5-P4)
3675   RETURN R3
3680   R3=R3-(Q5-R3)*P4/(P6-P4)
3685   RETURN R3
3690   R3=.5*(Q4+Q5)
3695   RETURN R3
3700   IF P4<0 THEN 3715
3705   R3=MIN(R3,R4)
3710   RETURN R3
3715   R3=MAX(R3,R4)
3720   RETURN R3
3725   E5=1      !R3 GIVEN AFTER 30 ITERATIONS.
3730   RETURN R3
3735   ! ***** A SUBROUTINE FOR P4*****
3740   P4=FNE11cov(R3,Hx,Hy,Hz,Sx,Sy,Sz,K0,K2,S)
3745   IF P4>0 THEN Q9=Q9+1
3750   I8=P4
3755   P4=P4-P3
3760   IF P4>0 THEN 3790
3765   Q4=R3
3770   I6=R3
3775   P5=P4
3780   I3=-P4
3785   RETURN

```

```

3790      Q5=R3
3795      I7=R3
3800      P6=P4
3805      I4=P4
3810      RETURN
3815  FNEND
3820  ! *****ELLCOV*****
3825  ! THIS PROGRAM IS CALLED "ELLCOV". (X,Y,Z) IS AN ELEMENT OF A CARTESIAN
3826  ! COORDINATE SYSTEM. ELLCOV RETURNS THE PROBABILITY OF A SHOT FALLING UNDER
3830  ! AN ELLIPSOIDAL NORMAL DISTRIBUTION, IN A SPHERE WITH CENTER (Hx,Hy,Hz)
3831  ! AND RADIUS R3. THE DISTRIBUTION HAS MEAN ZERO AND STANDARD DEVIATIONS Sx,
3835  ! Sy,Sz IN THE X,Y,Z DIRECTIONS, RESPECTIVELY. THE INPUT PARAMETERS ARE
3836  ! R3,Hx,Hy,Hz,Sx,Sy,Sz,K0,K2,S, WHERE K0=Hx*Hx+Hy*Hy+Hz*Hz, K2=SQR(K0)
3837  ! AND S=MAX(Sx,Sy,Sz).
3840  ! THE OUTPUT P IS GENERALLY ACCURATE TO AT LEAST 6-DECIMAL DIGITS.
3841  ! SUBROUTINES USED ARE CIRCV, DAWS, DXDAWS, DXERF, DXERF3, ELLCOV, EQSIG,
3845  ! ERF, ERF3, ERFC, ERFC1, FN.
3850  DEF FNEllcov(R3,Hx,Hy,Hz,Sx,Sy,Sz,K0,K2,S)
3855      COM X(43),Y(43),A6(10),B6(10),Q9,V,A7,P4,Ve,T6,P1,T9
3860      C5=6.02738      !5.6123
3865      C6=.398942280402      !1/SQR(2*PI)
3870      C7=.707106781187      !SQR(.5)
3875      C8=1.41421356237
3880      F2=4.262
3885      C10=.999999998334      !ERF(F2)
3890      B1=.564189583548      !1/SQR(PI)
3895      A4=.797884560803      !SQR(2/PI)
3900      Z4=.0000000025
3905      H=ABS(Hx)
3910      K=ABS(Hy)
3915      L=ABS(Hz)
3920      S1=Sx
3925      S2=Sy
3930      S3=Sz
3935      P4=0
3940      T5=0
3945      H1=1.5*Sx*Sy*Sz*Z4/C6
3950      K7=R3*R3*R3
3955      IF K7<=H1 THEN RETURN P4
3960      A3=(K2-R3)*C7/S
3965      A5=EXP(-A3*A3)
3970      IF K2<R3 THEN 3985
3975      IF K7*A5<=H1 THEN RETURN P4
3980      GOTO 4010
3985      P4=FNEqsig(R3,K0,K2,S,A3,A5,A4,C7) ! ELLCOV FOR EQUAL SIGMAS
3990      V=-3
3995      IF P4<1-Z4 THEN 4010
4000      P4=1
4005      RETURN P4
4010      IF (Sx<>Sy) AND (Sx<>Sz) AND (Sy<>Sz) THEN 4305
4015      IF (Sx<>Sy) OR (Sy<>Sz) THEN 4035
4020      IF R3>K2 THEN RETURN P4
4025      P4=FNEqsig(R3,K0,K2,S,A3,A5,A4,C7)
4030      RETURN P4

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4035 IF Sx=Sy THEN 4090
4040 IF Sx=Sz THEN 4070
4045 S3=Sx
4050 S1=Sz
4055 L=ABS(Hx)
4060 H=ABS(Hz)
4065 GOTO 4090
4070 S2=Sz
4075 S3=Sy
4080 K=ABS(Hz)
4085 L=ABS(Hy)
4090 IF H+K<>0 THEN 4280
4095 ! *****S1=S2 AND H=K=0*****
4100 L1=ABS((S1*S1-S3*S3)/(S1*S1))
4105 L2=SQR(L1)
4110 L4=(L+R3)*C7/S3
4115 L5=(L-R3)*C7/S3
4120 IF L5>F2 THEN RETURN P4
4125 W9=-1
4130 J3=EXP(-L5*L5)
4135 S0=C7/(S3*L2)
4140 IF L=0 THEN 4240
4145 IF R3<L THEN 4160
4150 P4=FNErf(L4)-FNErf3(L5,J3)
4155 GOTO 4170
4160 W9=EXP(-2*R3*L/(S3*S3))
4165 P4=J3*(FNErfc1(L5,1)-W9*FNErfc1(L4,1))
4170 F1=(L+L1*R3)*S0
4175 S0=(L-L1*R3)*S0
4180 IF S1<S3 THEN 4220
4185 V=-5
4190 IF S0<0 THEN 4210
4195 IF W9<0 THEN W9=EXP(-2*L*R3/(S3*S3))
4200 P4=.5*(P4-J3/L2*(FNErfc1(S0,1)-W9*FNErfc1(F1,1)))
4205 RETURN P4
4210 P4=.5*(P4-1/L2*EXP(-.5*(R3*R3-L*L/L1)/(S1*S1))*(FNErf(F1)-FNErf(S0)))
4215 RETURN P4
4220 V=-6
4225 IF W9<0 THEN W9=EXP(-2*L*R3/(S3*S3))
4230 P4=.5*P4-B1*J3/L2*(FNDaws(F1)-W9*FNDaws(S0))
4235 RETURN P4
4240 IF S3>S1 THEN 4260
4245 V=-5.5
4250 P4=FNErf3(L4,J3)-L4*EXP(-.5*R3*R3/(S1*S1))*FNDxerf3(L4*L2,-1)
4255 RETURN P4
4260 V=-6.5
4265 P4=FNErf3(L4,J3)-2*B1*J3*L4*FNDxdaws(L4*L2)
4270 RETURN P4
4275 ! *****S1=S2*****
4280 V=1
4285 D=SQR(H*H+K*K)
4295 GOTO 4415

```



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4300      ! *****H=K=0*****
4305      IF (H+K=0) AND (ABS(S1/S2-50000.000005)<49999.999995) THEN 4385
4310      IF (H+L=0) AND (ABS(S1/S3-50000.000005)<49999.999995) THEN 4355
4315      IF (K+L<>0) OR (ABS(S2/S3-50000.000005)>49999.999995) THEN 4510
4320      L1=S1
4325      S1=S3
4330      S3=L1
4335      L1=H
4340      H=L
4345      L=L1
4350      GOTO 4385
4355      L1=S2
4360      S2=S3
4365      S3=L1
4370      L1=K
4375      K=L
4380      L=L1
4385      IF S1>S2 THEN 4405
4390      W1=S1
4395      S1=S2
4400      S2=W1
4405      D=S2/S1
4410      V=0
4415      X1=C7/S3
4420      A5=X1*(L-R3)
4425      IF A5>F2 THEN RETURN P4
4430      W9=1
4435      IF A5>-F2 THEN 4475
4440      A5=-F2
4445      B5=F2
4450      X2=L*X1
4455      IF X2>=F2 THEN 4825
4460      W9=0
4465      B5=X2
4470      GOTO 4825
4475      B5=X1*L
4480      IF B5>F2 THEN 4495
4485      W9=0
4490      GOTO 4825
4495      B5=F2
4500      GOTO 4825
4505      ! *****GENERAL CASE*****
4510      V=2
4515      H(1)=H
4520      H(2)=K
4525      H(3)=L
4530      S(1)=S1
4535      S(2)=S2
4540      S(3)=S3
4545      ! *****FIX ORDER INTEGRATION*****
4550      T(1)=MAX(S1,S2)
4555      T(2)=MAX(S2,S3)
4560      T(3)=MAX(S3,S1)

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```

4565   W8(1)=S1/S2
4570   IF W8(1)>1 THEN W8(1)=1/W8(1)
4575   W8(2)=S2/S3
4580   IF W8(2)>1 THEN W8(2)=1/W8(2)
4585   W8(3)=S3/S1
4590   IF W8(3)>1 THEN W8(3)=1/W8(3)
4595   J=3
4600   IF W8(1)>W8(2) THEN 4645
4605   IF W8(1)<>W8(2) THEN 4620
4610   IF T(1)<T(2) THEN J=1
4615   GOTO 4670
4620   J=1
4625   IF W8(2)>W8(3) THEN 4670
4630   IF W8(2)<>W8(3) THEN 4660
4635   IF T(2)<T(3) THEN J=2
4640   GOTO 4670
4645   IF W8(1)>W8(3) THEN 4670
4650   IF W8(1)<>W8(3) THEN 4660
4655   IF T(1)>T(3) THEN 4670
4660   J=2
4665   ! *****FIND LIMITS OF INTEGRATION*****
4670   X1=C7/S(J)
4675   A5(J)=(H(J)-R3)*X1
4680   IF A5(J)>=F2 THEN RETURN P4
4685   W8(J)=1
4690   IF A5(J)>-F2 THEN 4730
4695   A5(J)=-F2
4700   B5(J)=F2
4705   X2=H(J)*X1
4710   IF X2>=F2 THEN 4755
4715   B5(J)=X2
4720   W8(J)=0
4725   GOTO 4755
4730   B5(J)=H(J)*X1
4735   IF B5(J)>F2 THEN 4750
4740   W8(J)=0
4745   GOTO 4755
4750   B5(J)=F2
4755   IF J=3 THEN 4810
4760   IF J=2 THEN 4790
4765   H=H(3)
4770   L=H(1)
4775   S1=S(3)
4780   S3=S(1)
4785   GOTO 4810
4790   L=H(2)
4795   S3=S(2)
4800   K=H(3)
4805   S2=S(3)
4810   A5=A5(J)
4815   B5=B5(J)
4820   W9=W8(J)

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4822  ! *****SPECIAL CASE*****
4825  Py=0
4830  IF A5>-F2 THEN 4915
4835  K7=L-C8*S3*B5
4840  T8=L-C8*S3*A5
4845  K7=R3*R3-K7*K7
4850  T8=R3*R3-T8*T8
4855  IF (K7<0) OR (T8<0) THEN 4915
4860  K7=SQR(K7)
4865  T8=SQR(T8)
4870  IF ABS(T8-K7)>K7*1E-8 THEN 4915
4875  V=-9+V
4880  IF V<-.7 THEN 4895
4885  P4=C10*FNEllcv(.5*(K7+T8),H,K,S1,S2)
4890  RETURN P4
4895  P4=C10*FNCircv(.5*(K7+T8),D,S1,V)
4900  RETURN P4
4910  ! *****START OF GAUSSIAN INTEGRATION*****
4915  K7=MAX(S1,S2)
4920  T8=C5*K7+SQR(H*H+K*K)
4925  T8=R3*R3-T8*T8
4930  IF T8<0 THEN 4985
4935  A3=SQR(T8)
4940  T8=(L-A3)/(C8*S3)
4945  IF T8>=B5 THEN 4985
4950  B5=T8
4955  K7=(L+A3)/(C8*S3)
4960  IF K7<4.4 THEN 4975
4965  Py=P4=.5*FNERfc(T8)
4970  GOTO 4985
4975  Py=P4=.5*(FNERf(K7)-FNERf(T8))
4980  IF (ABS(B5-A5)<=ABS(B5+A5)*1E-10) OR (B5<=A5) THEN RETURN P4
4985  A3=A5
4990  FOR I=-12 TO 12
4995    IF I=0 THEN 5020
5000    T5=.5*((B5-A5)*SGN(I)*X(31+ABS(I))+B5+A5)
5005    X2=FNFn(T5,R3,H,K,L,S1,S2,S3,V,D,W9)
5010    IF X2>1E-10 THEN 5025
5015    A3=T5
5020  NEXT I
5025  IF X2<1E-7 THEN 5115
5030  A5=A8=A3
5035  IF (T5-A5)*X2<=1E-9 THEN 5115
5040  T8=A5
5045  K7=(T5-A5)*(1-X(43))      !.5*K7=.5*(T5+A5-(T5-A5)*X(43))-A5
5050  IF (ABS(K7)<=ABS(A5)*1E-10) OR (K7<=0) THEN 5115
5055  T8=T8+K7
5060  X3=FNFn(T8,R3,H,K,L,S1,S2,S3,V,D,W9) ! COMPUTATION OF INTEGRAND
5065  IF X3>1E-10 THEN 5080
5070  A8=T8
5075  GOTO 5055
5080  A3=A5=A8
5085  IF X3<=1E-7 THEN 5115
5090  IF (T5-A5)*X2<=1E-9 THEN 5115

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5095 ! *****GAUSSIAN RULE. Pz GIVES THE INTEGRAL FROM
5096 !      A5 TO T5 OF THE INTEGRAND F(X)=FNFN(X).*****
5100 CALL Gquad(A5,T5,Ierr,1,Pz,E,12,N2,R3,H,K,L,S1,S2,S3,V,D,W9)
5105 A5=T5
5110 GOTO 5130
5115 Pz=0
5120 A5=A3
5125 ! *****GAUSSIAN RULE. P4 GIVES THE INTEGRAL FROM
5126 !      A5 TO B5 OF THE INTEGRAND F(X)=FNFN(X).*****
5130 CALL Gquad(A5,B5,Ierr,1,P4,E,12,N2,R3,H,K,L,S1,S2,S3,V,D,W9)
5135 P4=B1*(P4+Pz)+Py
5140 IF P4>1 THEN P4=1
5145 RETURN P4
5150 FNEND
5155 ! *****EQSIG*****
5160 ! FNEqsig COMPUTES ELLCOV FOR EQUAL SIGMAS.
5165 ! SUBROUTINES NEEDED: ERF, ERF3, ERFC1, REXP.
5170 DEF FNEqsig(R3,K0,K2,S,A3,A5,A4,C7)
5175 IF K0<>0 THEN 5195
5180 V=-3
5185 P4=FNErf3(-A3,A5)-A4*R3*A5/S
5190 RETURN P4
5195 K1=(R3+K2)*C7/S
5200 P5=2*R3*K2/(S*S)
5205 Q5=FNRexp(-P5)
5210 IF R3>K2 THEN 5230
5215 Q4=FNERfc1(K1,1)
5220 P4=A5*(FNERfc1(A3,1)-Q4+P5*Q5*Q4)
5225 GOTO 5235
5230 P4=FNErf(K1)-FNErf3(A3,A5)
5235 P4=.5*P4-A4*A5*R3*Q5/S
5240 V=-3.5
5245 RETURN P4
5250 FNEND
5255 ! *****GAMINV*****
5260 ! SUBPROGRAM FUNCTION GAMINV(A,X,X0,P,Q,IERR)
5265 ! INVERSE INCOMPLETE GAMMA RATIO FUNCTION
5270 ! GIVEN A>0 AND NONNEGATIVE P AND Q WHERE P+Q=1, THEN X IS OBTAINED
5271 ! WHERE P(A,X)=P AND Q(A,X)=Q.
5275 ! SCHRODER ITERATION IS EMPLOYED. THE ROUTINE ATTEMPTS TO COMPUTE X
5276 ! TO 10 SIGNIFICANT DIGITS.
5280 ! X IS A VARIABLE. IF P=0 THEN X IS ASSIGNED THE VALUE 0, AND IF Q=0
5281 ! THEN X IS SET TO THE LARGEST FLOATING POINT NUMBER AVAILABLE.
5285 ! OTHERWISE, GAMINV ATTEMPTS TO OBTAIN A SOLUTION FOR P(A,X)=P AND
5286 ! Q(A,X)=Q. IF THE ROUTINE IS SUCCESSFUL THE SOLUTION IS IN X.
5290 ! X0 IS AN OPTIONAL INITIAL APPROXIMATION FOR X. IF THE USER DOES NOT
5291 ! WISH TO GIVE AN INITIAL APPROXIMATION, THEN SET X0 <= 0.
5295 ! Ierr IS A VARIABLE THAT REPORTS THE STATUS OF THE RESULTS. WHEN THE
5296 ! ROUTINE TERMINATES, Ierr HAS ONE OF THE FOLLOWING VALUES:
5300 !      Ierr=0      THE SOLUTION WAS OBTAINED. ITERATION WAS NOT USED.
5305 !      Ierr>0     THE SOLUTION WAS OBTAINED. Ierr ITERATIONS WERE DONE.

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5310 !      Ierr=-2      (INPUT ERROR) A<=0.
5315 !      Ierr=-3      X IS TOO SMALL. X=0 GIVEN.
5320 !      Ierr=-4      (INPUT ERROR) P+Q # 1.
5325 !      Ierr=-6      20 ITERATIONS DONE. MOST RECENT VALUE OF X GIVEN.
5326 !                   CANNOT OCCUR IF X0 <= 0.
5330 !      Ierr=-7      ITERATION FAILED. NO VALUE FOR X IS GIVEN. THIS MAY
5331 !                   OCCUR WHEN X IS APPROXIMATELY 0.
5335 !      Ierr=-8      A VALUE FOR X HAS BEEN OBTAINED, BUT GAMINV IS NOT
5336 !                   CERTAIN OF ITS ACCURACY. ITERATION CANNOT BE DONE
5340 !                   IN THIS CASE. IF X0 <= 0, THIS CAN OCCUR ONLY WHEN P
5341 !                   OR Q IS APPROXIMATELY 0. IF X0 > 0 THEN THIS CAN OCCUR
5345 !                   WHEN A IS EXCEEDINGLY CLOSE TO X AND A IS VERY LARGE
5346 !                   (SAY A > 1E20).
5350 ! CODE TAKEN FROM THE FORTRAN VERSION BY AL MORRIS.
5355 ! SUBROUTINES NEEDED: GAMMAI, GRATIO, ALNREL, GAMLN, RCOMP, GAMLN1
5360 DEF FNGaminv(A,X,X0,P,Q,Ierr)
5365   C=.577215664902
5370   Ln10=2.302585
5375   A0=3.31125922109
5380   A1=11.6616720289
5385   A2=4.28342155967
5390   A3=.213623493716
5395   B1=6.61053765625
5400   B2=6.4069159776
5405   B3=1.27364489782
5410   B4=3.61170810188E-2
5415   Eps0(1)=1E-10
5420   Eps0(2)=1E-8
5425   Amin(1)=500
5430   Amin(2)=100
5435   Bmin(1)=1E-28
5440   Bmin(2)=1E-13
5445   Dmin(1)=1E-6
5450   Dmin(2)=1E-4
5455   Emin(1)=2E-3
5460   Emin(2)=6E-3
5465   Tol=1E-5
5470   E=1E-12
5475   Xmin=1E-99
5480   Xmax=9.999999999E99
5485   X=X0
5490   IF A<=0 THEN 6405
5495   T=-.5+P+Q-.5
5500   E2=E+E
5505   IF ABS(T)>10*E THEN 6425
5510   Ierr=0
5515   IF P=0 THEN RETURN X
5520   IF Q=0 THEN 6365
5525   IF A=1 THEN 6375
5530   Amax=4E-11/(E+E)
5535   Iop=1
5540   IF E>1E-10 THEN Iop=2
5545   Eps=Eps0(Iop)
5550   Xn=X0
5555   IF X0>0 THEN 6070

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5560 ! ****INITIAL APPROXIMATION Xn OF X FOR A<1.****
5565 IF A>1 THEN 5795
5570 G=1/FNGammai(.5+A+.5)
5575 Qg=Q*G
5580 IF Qg=0 THEN 6470
5585 B=Qg/A
5590 IF Qg>.6*A THEN 5735
5595 IF (A>=.3) OR (B<.35) THEN 5620
5600 T=EXP(-(B+C))
5605 U=T*EXP(T)
5610 Xn=T*EXP(U)
5615 GOTO 6070
5620 IF B>=.45 THEN 5735
5625 IF B=0 THEN 6470
5630 Y=-LOG(B)
5635 S=.5-A+.5
5640 Z=LOG(Y)
5645 T=Y-S*Z
5650 IF B<.15 THEN 5665
5655 Xn=Y-S*LOG(T)-LOG(1+S/(T+1))
5660 GOTO 6220
5665 IF B<=.01 THEN 5685
5670 U=((T+2*(3-A))*T+(2-A)*(3-A))/((T+(5-A))*T+2)
5675 Xn=Y-S*LOG(T)-LOG(U)
5680 GOTO 6220
5685 C1=-S*Z
5690 C2=-S*(1+C1)
5695 C3=S*((.5*C1+(2-A))*C1+(2.5-1.5*A))
5700 C4=-S*((C1/3+(2.5-1.5*A))*C1+((A-6)*A+7))*C1+((11*A-46)*A+47)/6)
5705 C5=-S*((C1/4+(11*A-17)/6)*C1+((-3*A+13)*A-13))*C1+.5*((2*A-25)*A+72)
5710 Xn=((C5/Y+C4)/Y+C3)/Y+C2)/Y+C1+Y
5715 IF A>1 THEN 6190
5720 IF B>Bmin(Iop) THEN 6220
5725 X=Xn
5730 RETURN X
5735 IF B*Q>1E-8 THEN 5750
5740 Xn=EXP(-(Q/A+C))
5745 GOTO 5770
5750 IF P<=.9 THEN 5765
5755 Xn=EXP((FNAlnrel(-Q)+FNGamln1(A))/A)
5760 GOTO 5770
5765 Xn=EXP(LOG(P*G)/A)
5770 IF Xn=0 THEN 6415
5775 T=.5+(.5-Xn/(.5+A+.5))
5780 Xn=Xn/T
5785 GOTO 6070
5790 ! *****INITIAL APPROXIMATION FOR A > 1.*****
5795 IF Q<=.5 THEN 5810
5800 W=LOG(P)
5805 GOTO 5815
5810 W=LOG(Q)
5815 T=SQR(-2*W)
5820 S=T-(((A3*T+A2)*T+A1)*T+A0)/((((B4*T+B3)*T+B2)*T+B1)*T+1)

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```

5825     IF Q>.5 THEN S=-S
5830     Rta=SQR(A)
5835     S2=S*S
5840     Xn=A+S*Rta+(S2-1)/3+S*(S2-7)/(36*Rta)-((3*S2+7)*S2-16)/(810*A)+S*((9*S2+
256)*S2-433)/(38880*A*Rta)
5845     Xn=MAX(Xn,0)
5850     IF A<Amin(Iop) THEN 5870
5855     X=Xn
5860     D=.5-X/A+.5
5865     IF ABS(D)<=Dmin(Iop) THEN RETURN X
5870     IF P<=.5 THEN 5940
5875     IF Xn<3*A THEN 6220
5880     Y=-(W+FNGamln(A))
5885     IF A<2 THEN 5920
5890     GOTO 6220
5895     D=MAX(2,A*(A-1))
5900     IF Y<Ln10*D THEN 5920
5905     S=.5-A+.5
5910     Z=LOG(Y)
5915     GOTO 5685
5920     T=-.5+A-.5
5925     Xn=Y+T*LOG(Xn)-FNAInrel(-T/(Xn+1))
5930     Xn=Y+T*LOG(Xn)-FNAInrel(-T/(Xn+1))
5935     GOTO 6220
5940     Ap1=A+1
5945     IF Xn>.7*Ap1 THEN 6075
5950     W=W+FNGamln(Ap1)
5955     IF Xn>.15*Ap1 THEN 6010
5960     Ap2=A+2
5965     Ap3=A+3
5970     X=EXP((W+X)/A)
5975     X=EXP((W+X-LOG(1+X/Ap1*(1+X/Ap2)))/A)
5980     X=EXP((W+X-LOG(1+X/Ap1*(1+X/Ap2)))/A)
5985     X=EXP((W+X-LOG(1+X/Ap1*(1+X/Ap2*(1+X/Ap3)))/A)
5990     Xn=X
5995     IF Xn>.01*Ap1 THEN 6010
6000     IF Xn<=Emin(Iop)*Ap1 THEN RETURN X
6005     GOTO 6075
6010     Apn=Ap1
6015     T=Xn/Apn
6020     Sum=1+T
6025     Apn=Apn+1
6030     T=T*Xn/Apn
6035     Sum=Sum+T
6040     IF T>1E-4 THEN 6025
6045     T=W-LOG(Sum)
6050     Xn=EXP((Xn+T)/A)
6055     Xn=Xn*(1-(A*LOG(Xn)-Xn-T)/(A-Xn))
6060     GOTO 6075
6065     ***SCHRODER ITERATION USING P--REF. VOL.1 OF HENRICI, 1974, (P.529),
6070     IF P>.5 THEN 6220
6075     IF P<=1E10*Xmin THEN 6455
6080     Am1=-.5+A-.5
6085     IF A<=Amax THEN 6100

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6090 D=.5-Xn/A+.5
6095 IF ABS(D)<=E2 THEN 6455
6100 IF Ierr>=20 THEN 6435
6105 Ierr=Ierr+1
6110 CALL Gratio(A,Xn,Pn,Qn,0)
6115 IF (Pn=0) OR (Qn=0) THEN 6455
6120 R=FNRcomp(A,Xn)
6125 IF R=0 THEN 6455
6130 T=(Pn-P)/R
6135 W=.5*(Am1-Xn)
6140 IF (ABS(T)<=.1) AND (ABS(W*T)<=.1) THEN 6165
6145 X=Xn*(1-T)
6150 IF X<=0 THEN 6445
6155 D=ABS(T)
6160 GOTO 6190
6165 H=T*(1+W*T)
6170 X=Xn*(1-H)
6175 IF X<=0 THEN 6445
6180 IF (ABS(W)>=1) AND (ABS(W)*T*T<=Eps) THEN RETURN X
6185 D=ABS(H)
6190 Xn=X
6195 IF D>Tol THEN 6085
6200 IF D<=Eps THEN RETURN X
6205 IF ABS(P-Pn)<=Tol*P THEN RETURN X
6210 GOTO 6085
6215 ! *****SCHRODER ITERATION USING Q*****
6220 IF Q<=1E10*Xmin THEN 6455
6225 Am1=-.5+A-.5
6230 IF A<=Amax THEN 6245
6235 D=.5-Xn/A+.5
6240 IF ABS(D)<=E2 THEN 6455
6245 IF Ierr>=20 THEN 6435
6250 Ierr=Ierr+1
6255 CALL Gratio(A,Xn,Pn,Qn,0)
6260 IF (Pn=0) OR (Qn=0) THEN 6455
6265 R=FNRcomp(A,Xn)
6270 IF R=0 THEN 6455
6275 T=(Q-Qn)/R
6280 W=.5*(Am1-Xn)
6285 IF (ABS(T)<=.1) AND (ABS(W*T)<=.1) THEN 6310
6290 X=Xn*(1-T)
6295 IF X<=0 THEN 6445
6300 D=ABS(T)
6305 GOTO 6335
6310 H=T*(1+W*T)
6315 X=Xn*(1-H)
6320 IF X<=0 THEN 6445
6325 IF (ABS(W)>=1) AND (ABS(W)*T*T<=Eps) THEN RETURN X
6330 D=ABS(H)
6335 Xn=X
6340 IF D>Tol THEN 6230
6345 IF D<=Eps THEN RETURN X
6350 IF ABS(Q-Qn)<=Tol*Q THEN RETURN X
6355 GOTO 6230

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6360 ! *****SPECIAL CASES*****
6365 X=Xmax
6370 RETURN X
6375 IF Q<.9 THEN 6390
6380 X=-FNAInrel(-P)
6385 RETURN X
6390 X=-LOG(Q)
6395 RETURN X
6400 ! *****ERROR RETURN*****
6405 Ierr=-2
6410 RETURN X
6415 Ierr=-3
6420 RETURN X
6425 Ierr=-4
6430 RETURN X
6435 Ierr=-6
6440 RETURN X
6445 Ierr=-7
6450 RETURN X
6455 X=Xn
6460 Ierr=-8
6465 RETURN X
6470 X=Xmax
6475 Ierr=-8
6480 FNEND
6485 ! *****GRATIO*****
6490 REM SUBROUTINES NEEDED: ERF,ERFC1, GAM1, GAMMAI, RLOG, REXP.
6495 SUB Gratio(A,X,Ans,Qans,Ind)
6500 DIM A(3),B(3),E(3),X(3)
6505 DIM W(20),D(13),D1(12),D2(10),D3(8),D4(6),D5(4),D6(2)
6510 E9=5E-12
6515 E=5E-13
6520 A(1)=5E-13
6525 A(2)=5E-7
6530 A(3)=5E-4
6535 B(1)=20
6540 B(2)=14
6545 B(3)=10
6550 E(1)=2.5E-4
6555 E(2)=.025
6560 E(3)=.14
6565 X(1)=31
6570 X(2)=17
6575 X(3)=9.7
6580 Ln10=2.30258509299
6585 Rpi=.398942280401
6590 Spi=1.77245385091
6595 Th=.333333333333
6600 D(1)=8.33333333333E-2
6605 D(2)=-1.48148148148E-2
6610 D(3)=1.15740740741E-3
6615 D(4)=3.52733686067E-4
6620 D(5)=-1.78755144033E-4
6625 D(6)=3.91926317852E-5

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6630 D(7)=-2.18544851068E-6  
6635 D(8)=-1.85406221072E-6  
6640 D(9)=8.29671134095E-7  
6645 D(10)=-1.76659527368E-7  
6650 D(11)=6.70785354340E-9  
6655 D(12)=1.02618097842E-8  
6660 D(13)=-4.38203601845E-9  
6665 D10=-1.85185185185E-3  
6670 D1(1)=-3.47222222222E-3  
6675 D1(2)=2.64550264550E-3  
6680 D1(3)=-9.90226337449E-4  
6685 D1(4)=2.05761316872E-4  
6690 D1(5)=-4.01877572016E-7  
6695 D1(6)=-1.80985503345E-5  
6700 D1(7)=7.64916091608E-6  
6705 D1(8)=-1.61209008946E-6  
6710 D1(9)=4.64712780281E-9  
6715 D1(10)=1.37863344692E-7  
6720 D1(11)=-5.75254560352E-8  
6725 D1(12)=1.19516285998E-8  
6730 D20=4.1335978836E-3  
6735 D2(1)=-2.68132716049E-3  
6740 D2(2)=7.71604938272E-4  
6745 D2(3)=2.00938786008E-6  
6750 D2(4)=-1.07366532264E-4  
6755 D2(5)=5.29234488291E-5  
6760 D2(6)=-1.27606351886E-5  
6765 D2(7)=3.4235787341E-8  
6770 D2(8)=1.37219573091E-6  
6775 D2(9)=-6.29899213838E-7  
6780 D2(10)=1.42806142061E-7  
6785 D30=6.49434156379E-4  
6790 D3(1)=2.29472093621E-4  
6795 D3(2)=-4.69189494395E-4  
6800 D3(3)=2.67720632063E-4  
6805 D3(4)=-7.56180167188E-5  
6810 D3(5)=-2.39650511387E-7  
6815 D3(6)=1.10826541153E-5  
6820 D3(7)=-5.67495282699E-6  
6825 D3(8)=1.42309007324E-6  
6830 D40=-8.61888290917E-4  
6835 D4(1)=7.8403922172E-4  
6840 D4(2)=-2.99072480303E-4  
6845 D4(3)=-1.46384525788E-6  
6850 D4(4)=6.64149821547E-5  
6855 D4(5)=-3.96836504718E-5  
6860 D4(6)=1.13757269707E-5  
6865 D50=-3.36798553366E-4  
6870 D5(1)=-6.97281375837E-5  
6875 D5(2)=2.77275324496E-4  
6880 D5(3)=-1.99325705162E-4  
6885 D5(4)=6.79778047794E-5  
6890 D60=5.31307936464E-4  
6895 D6(1)=-5.92166437354E-4  
6900 D6(2)=2.70878209672E-4  
6905 D70=3.44367606892E-4

```

6910 IF (A<0) OR (X<0) THEN 8055
6915 IF (A=0) AND (X=0) THEN 8055
6920 IF A*X=0 THEN 8020
6925 I=Ind+1
6930 IF (I<>1) AND (I<>2) THEN I=3
6935 Acc=MAX(A(I),E)
6940 E0=E(I)
6945 X0=X(I)
6950 ! *****SELECT APPROPRIATE ALGORITHM*****
6955 IF A>=1 THEN 6995
6960 IF A=.5 THEN 7980
6965 IF X<1.1 THEN 7370
6970 T1=A*LOG(X)-X
6975 U=A*EXP(T1)
6980 IF U=0 THEN 8040
6985 R=U*(1+FNGam1(A))
6990 GOTO 7605
6995 IF A>=B(I) THEN 7050
7000 IF (A>X) OR (X>=X0) THEN 7035
7005 Ta=A+A
7010 M=INT(Ta)
7015 IF Ta<>M THEN 7035
7020 J=INT(.5*M)
7025 IF A=J THEN 7510
7030 GOTO 7535
7035 T1=A*LOG(X)-X
7040 R=EXP(T1)*FNGammai(A)
7045 GOTO 7115
7050 L=X/A
7055 IF L=0 THEN 8025
7060 S=.5-L+.5
7065 Z=FNRlog(L)
7070 IF Z>=228/A THEN 8015
7075 Y=A*Z
7080 R1=SQR(A)
7085 IF ABS(S)<=E0/R1 THEN 7830
7090 IF ABS(S)<=.4 THEN 7700
7095 T=1/(A*A)
7100 T1=(((.75*T-1)*T+3.5)*T-105)/(A*1260)
7105 T1=T1-Y
7110 R=Rpi*R1*EXP(T1)
7115 IF R=0 THEN 8020
7120 IF X<=MAX(A,Ln10) THEN 7140
7125 IF X<X0 THEN 7605
7130 GOTO 7255
7135 ! *****TAYLOR SERIES FOR P/R*****
7140 Apn=A+1
7145 T=X/Apn
7150 W(1)=T
7155 FOR N=2 TO 20
7160 Apn=Apn+1
7165 T=T*(X/Apn)
7170 IF T<=1E-3 THEN 7190
7175 W(N)=T
7180 NEXT N

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```

7185     N=20
7190     S=T
7195     Tol=.5*Acc
7200     Apn=Apn+1
7205     T=T*(X/Apn)
7210     S=S+T
7215     IF T>Tol THEN 7200
7220     FOR M1=N-1 TO 1 STEP -1
7225         S=S+W(M1)
7230     NEXT M1
7235     Ans=R/A*(1+S)
7240     Qans=.5-Ans+.5
7245     SUBEXIT
7250     ! *****ASYMPTOTIC EXPANSION*****
7255     A1=A-1
7260     T=A1/X
7265     W(1)=T
7270     FOR N=2 TO 20
7275         A1=A1-1
7280         T=T*(A1/X)
7285         IF ABS(T)<=1E-3 THEN 7305
7290         W(N)=T
7295     NEXT N
7300     N=20
7305     S=T
7310     IF ABS(T)<=Acc THEN 7335
7315     A1=A1-1
7320     T=T*(A1/X)
7325     S=S+T
7330     GOTO 7310
7335     FOR M=N-1 TO 1 STEP -1
7340         S=S+W(M)
7345     NEXT M
7350     Qans=R/X*(1+S)
7355     Ans=.5-Qans+.5
7360     SUBEXIT
7365     ! *****TAYLOR SERIES FOR P(A,X)/X**A*****
7370     A1=3
7375     C=X
7380     S=X/(A+3)
7385     Tol=3*Acc/(A+1)
7390     A1=A1+1
7395     C=-C*(X/A1)
7400     T=C/(A+A1)
7405     S=S+T
7410     IF ABS(T)>Tol THEN 7390
7415     J=A*X*((S/6-.5/(A+2))*X+1/(A+1))
7420     Z=A*LOG(X)
7425     H=FNGam1(A)
7430     G=1+H
7435     IF X<.25 THEN 7450
7440     IF A<X/2.59 THEN 7475
7445     GOTO 7455
7450     IF Z>-.13394 THEN 7475

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```

7455 W=EXP(Z)
7460 Ans=W*G*(.5-J+.5)
7465 Qans=.5-Ans+.5
7470 SUBEXIT
7475 L=FNRexp(Z)
7480 W=.5+L+.5
7485 Qans=(W*J-L*Z)*G-H
7490 IF Qans<0 THEN 8040
7495 Ans=.5-Qans+.5
7500 SUBEXIT
7505 ! *****FINITE SUMS FOR Q WHEN A>=1 AND 2*A IS AN INTEGER.*****
7510 S=EXP(-X)
7515 T=S
7520 K=1
7525 C=0
7530 GOTO 7560
7535 T1=SQR(X)
7540 S=FNERfc1(T1,0)
7545 T=EXP(-X)/(Spi*T1)
7550 K=0
7555 C=-.5
7560 FOR N=K TO J-1
7565 C=C+1
7570 T=X*T/C
7575 S=S+T
7580 NEXT N
7585 Qans=S
7590 Ans=.5-Qans+.5
7595 SUBEXIT
7600 ! *****CONTINUED FRACTION EXPANSION*****
7605 Tol=MAX(5*E9,Acc)
7610 A21=1
7615 A2=1
7620 B21=X
7625 B2=X+.5-A+.5
7630 C=1
7635 A21=X*A2+C*A21
7640 B21=X*B2+C*B21
7645 Amo=A21/B21
7650 C=C+1
7655 C1=C-A
7660 A2=A21+C1*A2
7665 B2=B21+C1*B2
7670 Ano=A2/B2
7675 IF ABS(Ano-Amo)>=Tol*Ano THEN 7635
7680 Qans=R*Ano
7685 Ans=.5-Qans+.5
7690 SUBEXIT
7695 ! *****GENERAL TEMME EXPNSION*****
7700 IF (ABS(S)<=2*E9) AND (A*E9*E9>3.28E-3) THEN 8055
7705 C=EXP(-Y)
7710 W=.5*FNERfc1(SQR(Y),1)!*****
7715 U=1/A
7720 Z=SQR(Z+Z)
7725 IF L<1 THEN Z=-Z
7730 IF I>=2 THEN 7785

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```

7735 IF ABS(S)<=1E-3 THEN 7865
7740 C0=(((((D(13)*Z+D(12))*Z+D(11))*Z+D(10))*Z+D(9))*Z+D(8))*Z+D(7))*
Z+D(6))*Z+D(5))*Z+D(4))*Z+D(3))*Z+D(2))*Z+D(1))*Z-Th
7745 C1=(((((D1(12)*Z+D1(11))*Z+D1(10))*Z+D1(9))*Z+D1(8))*Z+D1(7))*Z+D1
(6))*Z+D1(5))*Z+D1(4))*Z+D1(3))*Z+D1(2))*Z+D1(1))*Z+D10
7750 C2=(((((D2(10)*Z+D2(9))*Z+D2(8))*Z+D2(7))*Z+D2(6))*Z+D2(5))*Z+D2(4))*
Z+D2(3))*Z+D2(2))*Z+D2(1))*Z+D20
7755 C3=(((((D3(8)*Z+D3(7))*Z+D3(6))*Z+D3(5))*Z+D3(4))*Z+D3(3))*Z+D3(2))*Z+
D3(1))*Z+D30
7760 C4=(((((D4(6)*Z+D4(5))*Z+D4(4))*Z+D4(3))*Z+D4(2))*Z+D4(1))*Z+D40
7765 C5=(((((D5(4)*Z+D4(3))*Z+D5(2))*Z+D5(1))*Z+D50
7770 C6=(((((D6(2)*Z+D6(1))*Z+D60
7775 T=(((((D70*U+C6)*U+C5)*U+C4)*U+C3)*U+C2)*U+C1)*U+C0
7780 GOTO 7940
7785 IF I>2 THEN 7815
7790 C0=(((((D(6)*Z+D(5))*Z+D(4))*Z+D(3))*Z+D(2))*Z+D(1))*Z-Th
7795 C1=(((((D1(4)*Z+D1(3))*Z+D1(2))*Z+D1(1))*Z+D10
7800 C2=D2(1)*Z+D20
7805 T=(C2*U+C1)*U+C0
7810 GOTO 7940
7815 T=((D(3)*Z+D(2))*Z+D(1))*Z-Th
7820 GOTO 7940
7825 ! *****TEMME EXPANSION FOR L=1*****
7830 IF A*E9*E9>3.28E-3 THEN 8055
7835 C=.5-Y+.5
7840 W=(.5-SQR(Y)*(1.5+(.5-Y/3))/Spi)/C
7845 U=1/A
7850 Z=SQR(Z+Z)
7855 IF L<1 THEN Z=-Z
7860 IF I>=2 THEN 7910
7865 C0=(((((D(7)*Z+D(6))*Z+D(5))*Z+D(4))*Z+D(3))*Z+D(2))*Z+D(1))*Z-Th
7870 C1=(((((D1(6)*Z+D1(5))*Z+D1(4))*Z+D1(3))*Z+D1(2))*Z+D1(1))*Z+D10
7875 C2=(((((D2(5)*Z+D2(4))*Z+D2(3))*Z+D2(2))*Z+D2(1))*Z+D20
7880 C3=(((((D3(4)*Z+D3(3))*Z+D3(2))*Z+D3(1))*Z+D30
7885 C4=(((((D4(2)*Z+D4(1))*Z+D40
7890 C5=(((((D5(2)*Z+D5(1))*Z+D50
7895 C6=D6(1)*Z+D60
7900 T=(((((D70*U+C6)*U+C5)*U+C4)*U+C3)*U+C2)*U+C1)*U+C0
7905 GOTO 7940
7910 IF I>2 THEN 7935
7915 C0=(D(2)*Z+D(1))*Z+C0
7920 C1=D1(1)*Z+D10
7925 T=(D20*U+C1)*U+C0
7930 GOTO 7940
7935 T=D(1)*Z-Th
7940 IF L<1 THEN 7960
7945 Qans=C*(W+Rpi*T/R1)
7950 Ans=.5-Qans+.5
7955 SUBEXIT
7960 Ans=C*(W-Rpi*T/R1)
7965 Qans=.5-Ans+.5
7970 SUBEXIT

```

```

7975 ! *****SPECIAL CASES*****
7980 IF X>=.25 THEN 8000
7985 Ans=FNerf(SQR(X))
7990 Qans=.5-Ans+.5
7995 SUBEXIT
8000 Qans=FNerfc1(SQR(X),0)
8005 Ans=.5-Qans+.5
8010 SUBEXIT
8015 IF ABS(S)<=2*E9 THEN 8055
8020 IF X>A THEN 8040
8025 Ans=0
8030 Qans=1
8035 SUBEXIT
8040 Ans=1
8045 Qans=0
8050 SUBEXIT
8055 Ans=2
8060 SUBEND
8065 ! *****ALNREL*****
8070 DEF FNAlnrel(A)
8075 P1=-1.29418923022E0
8080 P2=.405303492862
8085 P3=-1.78874546012E-2
8090 Q1=-1.62752256355E0
8095 Q2=.747811014038
8100 Q3=-8.45104217946E-2
8105 IF ABS(A)>.375 THEN 8130
8110 T=A/(A+2)
8115 T2=T*T
8120 W=(((P3*T2+P2)*T2+P1)*T2+1)/(((Q3*T2+Q2)*T2+Q1)*T2+1)
8125 RETURN 2*T*W
8130 X=.5+A+.5
8135 RETURN LOG(X)
8140 FNEND
8145 ! *****GAMMAI*****
8150 ! Gammai=1/(GAMMA FUNCTION OF A). ! NEEDS GAM1.
8155 DEF FNGammai(A)
8160 IF A>=20 THEN 8245
8165 E=5E-13
8170 I=INT(A)
8175 I1=A-I
8180 Gammai=1
8185 IF I1<>0 THEN 8200
8190 I6=0
8195 GOTO 8220
8200 I6=-.5+I1-.5
8205 H=FNGam1(I1) ! I6=I1-.5-.5
8210 Gammai=I1+I1*H
8215 IF I=0 THEN RETURN Gammai
8220 I2=I-1
8225 I6=I6+1
8230 Gammai=Gammai/I6
8235 IF I6<I2 THEN 8225
8240 RETURN Gammai

```

```

8245     IF A<69.5 THEN 8260
8250     Gammai=0
8255     RETURN Gammai
8260     B=.918938533205
8265     I1=1/A
8270     I2=I1*I1
8275     I6=(.5-A)*LOG(A)+A-B-((1/1260*I2-1/360)*I2+1/12)*I1
8280     Gammai=EXP(I6)
8285     RETURN Gammai
8290  FNEND
8295  ! *****RCOMP*****
8300  ! COMPUTATION OF X^A*EXP(-X)/GAMMA(A)
8305  ! SUBROUTINES NEEDED: GAM1, GAMMAI, RLOG.
8310  DEF FNRcomp(A,X)
8315     Rt2pin=.398942280401 !1/SQR(2*PI)
8320     Rcomp=0
8325     IF A>=20 THEN 8360
8330     T=A*LOG(X)-X
8335     IF A>=1 THEN 8350
8340     Rcomp=A*EXP(T)*((1+FNGam1(A))
8345     RETURN Rcomp
8350     Rcomp=EXP(T)*FNGammai(A)
8355     RETURN Rcomp
8360     U=X/A
8365     IF U=0 THEN RETURN Rcomp
8370     T=1/(A*A)
8375     T1=(((.75*T-1)*T+3.5)*T-105)/(1260*A)
8380     T1=T1-A*FNRlog(U)
8385     Rcomp=Rt2pin*SQR(A)*EXP(T1)
8390     RETURN Rcomp
8395  FNEND
8400  ! *****COMPUTES Gamln(My)=LOG(GAMMA(My))*****
8405  ! NEEDS GAMLN1
8410  DEF FNGamln(My)
8415     J5=.418938533205 !J5=.5*LOG(2*PI-1)
8420     J0=8.33333333333E-2
8425     J1=-2.77777777777E-3
8430     J2=7.93650663184E-4
8435     J3=-5.95156336429E-4
8440     J4=8.20756370354E-4
8445     IF My>.8 THEN 8460 !Lg(My)
8450     Gamln=FNGamln1(My)-LOG(My)!SUBROUTINE FOR LOG(GAMMA(My+1))
8455     RETURN Gamln
8460     IF My>=2.25 THEN 8480
8465     Ny=-.5+My-.5
8470     Gamln=FNGamln1(Ny)
8475     RETURN Gamln
8480     IF My>=15 THEN 8535
8485     Ny=INT(My-1.25)
8490     Ay=My
8495     Wy=1

```



```

8500   FOR Iy=1 TO Ny
8505     Ay=Ay-1
8510     Wy=Ay*Wy
8515   NEXT Iy
8520   Ayy=Ay-1
8525   Gamln=FNGamln1(Ayy)+LOG(Wy)
8530   RETURN Gamln
8535   Ayy=1/(My*My)
8540   Wy=((((J4*Ayy+J3)*Ayy+J2)*Ayy+J1)*Ayy+J0)/My
8545   Gamln=J5+Wy+(My-.5)*(LOG(My)-1)
8550   RETURN Gamln
8555   FNEND
8560   ! *****COMPUTES Rlog=L-1+LOG(L) FOR L>0.*****
8565   DEF FNRlog(L)
8570     B3=.333333333333
8575     F2=.2
8580     F3=1.42857142857E-1
8585     F4=1.11111111111E-1
8590     F5=.896969696969E-2
8595     F6=5.88434334173E-2   !**F6=.695-1-LOG(.695).**
8600     F7=4.56512608816E-2   !**F7=4/3-1-LOG(4/3).**
8605     S1=.5-L+.5
8610     IF (L<.548) OR (L>1.681) THEN 8685   !**SUBROUTINE FOR R2=L-1-LOG(L).**
8615     IF L<.83667 THEN 8640
8620     IF L>1.1547 THEN 8655
8625     R2=-S1
8630     W1=0
8635     GOTO 8665
8640     R2=(L-.695)/.695
8645     W1=F6-R2*.305
8650     GOTO 8665
8655     R2=-.75*S1-.25
8660     W1=F7+R2*B3
8665     R2=R2/(R2+2)
8670     T=R2*R2
8675     Rlog=2*T*(1/(.5-R2+.5)-R2*(((F5*T+F4)*T+F3)*T+F2)*T+B3))+W1
8680     RETURN Rlog
8685     Rlog=-S1-LOG(L)
8690     RETURN Rlog
8695   FNEND
8700   ! *****GAM1=1/GAMMA(A+1)-1, -.5<=A<=1.5.*****
8705   DEF FNGam1(A)
8710     DIM P(7),Q(5),R(9)
8715     P(1)=.577215664902
8720     P(2)=-.409078193006
8725     P(3)=-.230975380858
8730     P(4)=5.97275330452E-2
8735     P(5)=7.66968181649E-3
8740     P(6)=-5.14889771324E-3
8745     P(7)=5.89597428611E-4
8750     Q(1)=1
8755     Q(2)=.427569613095
8760     Q(3)=.158451672430
8765     Q(4)=2.61132021441E-2

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8770      Q(5)=4.23244297897E-3
8775      R(1)=-.422784335098
8780      R(2)=-.771330383816
8785      R(3)=-.244757765222
8790      R(4)=.118378989872
8795      R(5)=9.30357293360E-4
8800      R(6)=-1.18290993445E-2
8805      R(7)=2.23047661158E-3
8810      R(8)=2.66505979059E-4
8815      R(9)=-1.32674909766E-4
8820      S1=.273076135304
8825      S2=5.59398236957E-2
8830      T=A
8835      D=A-.5
8840      IF D>0 THEN T=D-.5
8845      IF T<>0 THEN 8860
8850      Gam1=0
8855      RETURN Gam1
8860      IF T<0 THEN 8895
8865      Gam1=((((((P(7)*T+P(6))*T+P(5))*T+P(4))*T+P(3))*T+P(2))*T+P(1))/((((Q(5)
*T+Q(4))*T+Q(3))*T+Q(2))*T+Q(1))
8870      IF D>0 THEN 8885
8875      Gam1=A*Gam1
8880      RETURN Gam1
8885      Gam1=T/A*(-.5+Gam1-.5)
8890      RETURN Gam1
8895      Gam1=((((((((R(9)*T+R(8))*T+R(7))*T+R(6))*T+R(5))*T+R(4))*T+R(3))*T+R(2)
)*T+R(1))/((S2*T+S1)*T+1)
8900      IF D>0 THEN 8915
8905      Gam1=A*(.5+Gam1+.5)
8910      RETURN Gam1
8915      Gam1=T*Gam1/A
8920      RETURN Gam1
8925      FNEND
8930      ! *****REXP=(EXP(X)-1)/X*****
8935      DEF FNRExp(X)
8940      P1=9.1404191482E-10
8945      P2=2.3808236104E-2
8950      Q1=-.499999999086
8955      Q2=.107141568981
8960      Q3=-1.19041179761E-2
8965      Q4=5.95130811186E-4
8970      IF ABS(X)>.15 THEN 8985
8975      Rexp=((P2*X+P1)*X+1)/((((Q4*X+Q3)*X+Q2)*X+Q1)*X+1)
8980      RETURN Rexp
8985      W=EXP(X)
8990      IF X>0 THEN 9005
8995      Rexp=(W-.5-.5)/X
9000      RETURN Rexp
9005      Rexp=W*(.5+(.5-1/W))/X
9010      RETURN Rexp
9015      FNEND

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9020 ! *****ERF(X)*****
9025 DEF FNERf(X)
9030   X2=X*X
9035   IF ABS(X)>.5 THEN 9050
9040   Erf=X*((( -3.56098437018E-2*X2+6.99638348862)*X2+21.9792616183)*X2+242.66
7955231)/(((X2+15.0827976304)*X2+91.1649054045)*X2+215.05887587)
9045   RETURN Erf
9050   K4=ABS(X)
9055   IF K4>4 THEN 9085
9060   K3=((( -6.08581519597E-6*K4+.564371606864)*K4+4.26772010709)*K4+14.57189
85969)*K4+26.0947469561)*K4+22.8989928517
9065   K3=K3/(((K4+7.56884822936)*K4+26.2887957588)*K4+50.2732028638)*K4+51.9
335706876)*K4+22.8989857499)
9070   Erf=1-EXP(-X2)*K3
9075   IF X<0 THEN RETURN -Erf
9080   RETURN Erf
9085   Erf=1
9090   IF K4>=4.883 THEN 9115
9095   K6=1/X2
9100   B1=.564189583548! 1/SQR(PI)
9105   Erf=(B1-K6*(((3.24319519278E-2*K6+.243911029489)*K6+.119903955268)*K6+.0
12130827639)/((K6+1.43771227937)*K6+.489552441961)*K6+4.30026643453E-2))/K4
9110   Erf=1-EXP(-X2)*Erf
9115   IF X<0 THEN RETURN -Erf
9120   RETURN Erf
9125 FNEND
9130 ! *****ERFC(X)*****
9135 ! IF Ind=0 THEN Erfc1=ERFC(X); ELSE Erfc1=EXP(X*X)*ERFC(X).
9140 DEF FNERfc1(X,Ind)
9145   X2=X*X
9150   IF ABS(X)>.5 THEN 9170
9155   Erfc1=.5-X*((( -3.56098437018E-2*X2+6.99638348862)*X2+21.9792616183)*X2+2
42.667955231)/(((X2+15.0827976304)*X2+91.1649054045)*X2+215.05887587)+.5
9160   IF Ind>0 THEN RETURN EXP(X2)*Erfc1
9165   RETURN Erfc1
9170   K4=ABS(X)
9175   IF K4>4 THEN 9205
9180   K3=((( -6.08581519597E-6*K4+.564371606864)*K4+4.26772010709)*K4+14.57189
85969)*K4+26.0947469561)*K4+22.8989928517
9185   Erfc1=K3/(((K4+7.56884822936)*K4+26.2887957588)*K4+50.2732028638)*K4+5
1.9335706876)*K4+22.8989857499)
9190   IF Ind=0 THEN 9245
9195   IF X<0 THEN RETURN 2*EXP(X2)-Erfc1
9200   RETURN Erfc1
9205   IF X<-4.89 THEN 9260
9210   IF (X)=14.9891301) AND (Ind=0) THEN RETURN 0
9215   K6=1/X2
9220   B1=.564189583548! 1/SQR(PI)
9225   Erfc1=(B1-K6*(((3.24319519278E-2*K6+.243911029489)*K6+.119903955268)*K6+
.012130827639)/((K6+1.43771227937)*K6+.489552441961)*K6+4.30026643453E-2))/K4
9230   IF Ind=0 THEN 9245
9235   IF X<0 THEN RETURN 2*EXP(X2)-Erfc1
9240   RETURN Erfc1
9245   Erfc1=EXP(-X2)*Erfc1
9250   IF X<0 THEN RETURN 2-Erfc1
9255   RETURN Erfc1

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9260 IF Ind<>0 THEN RETURN 2*EXP(X2)
9265 RETURN 2
9270 FNEND
9275 ! *****COMPUTES Gamln1=LOG(GAMMA(1+Ayy), -.2<=Ayy<=1.25*****
9280 DEF FNGamln1(Ayy)
9285 G0=.577215664902 !SUBROUTINE FOR LOGGAMMA(My)=Lg
9290 G1=.844203922187
9295 G2=-.168860593647
9300 G3=-.780427615534
9305 G4=-.402055799310
9310 G5=-6.73562214326E-2
9315 G6=-2.71935708323E-3
9320 H1=2.88743195474E0
9325 H2=3.12755088915E0
9330 H3=1.56875193295E0
9335 H4=.361951990101
9340 H5=3.25038868254E-2
9345 H6=6.67465618796E-4
9350 M0=.422784335098
9355 M1=.848044614535
9360 M2=.565221050692
9365 M3=.156513060487
9370 M4=1.70502484023E-2
9375 M5=4.97958207639E-4
9380 V1=1.24313399878E0
9385 V2=.548042109832
9390 V3=.10155218744
9395 V4=7.13309612391E-3
9400 V5=1.1616547599E-4
9405 IF Ayy>=.60 THEN 9425
9410 Gamln1=(G6*Ayy+G5)*Ayy+G4)*Ayy+G3)*Ayy+G2)*Ayy+G1)*Ayy+G0)/((((H6
*Ayy+H5)*Ayy+H4)*Ayy+H3)*Ayy+H2)*Ayy+H1)*Ayy+1)
9415 Gamln1=-Ayy*Gamln1
9420 RETURN Gamln1
9425 Ny=Ayy-.5-.5
9430 Gamln1=(M5*Ny+M4)*Ny+M3)*Ny+M2)*Ny+M1)*Ny+M0)/((((V5*Ny+V4)*Ny+V3)*
Ny+V2)*Ny+V1)*Ny+1)
9435 Gamln1=Ny*Gamln1
9440 RETURN Gamln1
9445 FNEND
9450 ! *****DAWS(X)*****
9455 ! Daws(X) = EXP(-X*X)* INTEGRAL FROM 0 TO X OF EXP(T*T) DT.
9456 ! REF: MATH OF COMP. 1970, PP. 171-178. DAWSON'S INTEGRAL.
9460 DEF FNDaws(K1)
9465 Daws=K1*K1
9470 IF ABS(K1)>2.5 THEN 9490
9475 K4=(K1*(K1+2.38594565696E-2)*Daws+1.67795116189)*Daws-15.1982152422)*Daws+41
9.67290228)*Daws-1284.05832279)*Daws+10832.6558873
9480 Daws=K1*K4/((Daws+19.9422336364)*Daws+219.728331833)*Daws+1489.435572
42)*Daws+5937.71276935)*Daws+10832.6558772)
9485 RETURN Daws
9490 IF ABS(K1)>3.5 THEN 9520
9495 K4=88.7619386764/(Daws+3.47393742586)
9500 K4=13.8216341182/(Daws-12.6817901598+K4)

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9505 K4=37.3902763512/(Daws+4.07068101667+K4)
9510 Daws=(.214221965778/(Daws-4.91605365741+K4)+.500652754437)/K1
9515 RETURN Daws
9520 IF ABS(K1)>=5 THEN 9545
9525 K4=17.761778077/(Daws-4.1083233770)
9530 K4=-.53714272981/(Daws-10.385024821+K4)
9535 Daws=(.24924632421/(Daws-1.5853935006+K4)+.50000965199)/K1
9540 RETURN Daws
9545 K4=-4.12544065608/(Daws-11.1952164237)
9550 K4=-2.48787658804/(Daws-4.67312022141+K4)
9555 Daws=.5*(1+(.749999190567/(Daws-2.50017116686+K4)+.500000001674)/Daws)/K
1
9560 RETURN Daws
9565 FNEND
9570 ! *****ELLCV*****
9575 ! THIS PROGRAM IS CALLED "ELLCV". IT SUPPLIES
9576 ! THE ELLIPTICAL COVERAGE FUNCTION: E11cv=P(R,H,K,S1,S2).
9580 ! E11cv DENOTES THE PROBABILITY OF A SHOT, NORMALLY DISTRIBUTED WITH
9585 ! MEAN (0,0) AND STANDARD DEVIATIONS S1,S2 IN THE X AND Y DIRECTIONS,
9590 ! RESPECTIVELY, FALLING IN A CIRCLE IN THE XY-PLANE OF RADIUS R AND
9591 ! CENTERED AT (H,K). THE INPUT IS R,H,K,S1,S2. THE OUTPUT IS E11cv.
9595 ! PROGRAM IS SET FOR 8-DECIMAL-DIGIT ACCURACY IN E11cv.
9600 ! ELLCV USES ERFC, ERF WITH 12 DIGIT RELATIVE ACCURACY.
9605 ! SOURCES: NWL REPORT #1710, AUG.1960. MATH OF COMP. OCT. 1961,
9606 ! PP. 375,382. NSWC REPORT #83-13, NOVEMBER, 1982.
9610 DEF FNE11cv(R,H,K,S1,S2)
9615 COM X(*),Y(*)
9620 B1=.564189583548! 1/SQR(PI)
9625 Z3=.000000005*S1*S2
9630 IF R*R<=Z3 THEN RETURN 0
9635 A=6.027 !6 !5.6123
9640 B=1.41421356237 ! SQR(2)
9645 B2=36 !31.49791129 !B2=A*A
9650 H2=H*H+K*K
9655 H8=ABS(H)
9660 K8=ABS(K)
9665 D=MAX(S1,S2)
9670 T=R-A*D
9675 ! ***PROCEED TO SEE IF P=0 OR P=1
9680 IF T<0 THEN 9695
9685 IF T*T<H2 THEN 9695
9690 RETURN 1
9695 IF R-H8+A*S1<=0 THEN RETURN 0
9700 IF R-K8+A*S2<=0 THEN RETURN 0
9705 S0=SQR(H2)
9710 IF S0<=R THEN 9730
9715 D=(S0-R)*(S0-R)/(D*D)
9720 IF R*R*EXP(-.5*D)>Z3 THEN 9730
9725 RETURN 0
9730 IF S1<>S2 THEN 9750
9735 H8=S0
9740 K8=0
9745 IF R-H8+A*S1<=0 THEN RETURN 0
9750 IF H8*K8=0 THEN 9855

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9755   H9=H8/S1
9760   K9=K8/S2
9765   D=H9*H9+K9*K9
9770   Z2=R/S2
9775   IF D<=B2 THEN 9855
9780   Z1=R+A*MIN(S1,S2)
9785   IF H2<Z1*Z1 THEN 9855
9790   Q=S2/S1
9795   Q1=Q*Q
9800   F=Q1*H9*H9+K9*K9
9805   Z1=Z2*Z2*F/D
9810   Z=D-Z1-B2
9815   IF Z<0 THEN 9825
9820   IF Z*Z-4*Z1*B2>=0 THEN RETURN 0
9825   T1=H8*H8+Q1*K8*K8
9830   Z8=B2*S1*S1*T1/H2
9835   R2=R*R
9840   Y=H2-R2-Z8
9845   IF Y<0 THEN 9855
9850   IF Y*Y-4*R2*Z8>=0 THEN RETURN 0
9855   Z8=0      !****FIND LIMITS OF INTEGRATION
9860   Z=K8+A*S2
9865   H3=K8-A*S2
9870   S0=S1
9875   S9=S2
9880   Z=R-Z
9885   H5=0
9890   D1=0
9895   IF Z>=0 THEN D1=SQR(Z/R)
9900   IF H3>=0 THEN 9915
9905   E3=1-D1
9910   GOTO 9925
9915   E3=SQR(1-H3/R)-D1
9920   H5=1
9925   IF Z8<>0 THEN 9965
9930   Z8=1
9935   F=E3
9940   T=D1
9945   Z=H8+A*S1
9950   H6=H5
9955   H3=H8-A*S1
9960   GOTO 9880
9965   IF F>=E3 THEN 10010
9970   E3=F
9975   D1=T
9980   S9=S1
9985   Z8=H8
9990   S0=S2
9995   H8=K8
10000  K8=Z8
10005  H5=H6
10010  E3=.5*E3  !***BEGIN GAUSSIAN INTEGRATION****
10015  N=3
10020  IF (H8-R)>-3.5*S0 OR (K8-R)>-3.5*S9 THEN N=E3*R*(.34/S0+1/(.025*ABS(R-K8)
+5*S9))

```

```

10025  Z2=R/(B*S9)
10030  R8=R/(B*S0)
10035  H8=H8/(B*S0)
10040  K8=K8/(B*S9)
10045  IF N<2.75 THEN 10065
10050  J=31
10055  N1=12
10060  GOTO 10155
10065  IF N<1.35 THEN 10085
10070  J=21
10075  N1=10
10080  GOTO 10155
10085  IF N<.75 THEN 10105
10090  J=13
10095  N1=8
10100  GOTO 10155
10105  IF N<.35 THEN 10125
10110  J=7
10115  N1=6
10120  GOTO 10155
10125  IF N<.15 THEN 10145
10130  J=3
10135  N1=4
10140  GOTO 10155
10145  J=0
10150  N1=3
10155  Z=0
10160  Y=B1*E3*R8
10165  K9=0      !1.04E-8
10170  H9=2      !1.9999999702
10175  G3=0
10180  IF K8=0 THEN 10390
10185  Z3=0
10190  FOR I=-N1 TO N1
10195      IF I=0 THEN 10370
10200      T=E3*(SGN(I)*X(J+ABS(I))+1)+D1
10205      T9=T*T
10210      T1=R8*(1-T9)
10215      T2=(H8-T1)*(H8-T1)
10220      T4=EXP(-T2)
10225      IF H8<>0 THEN 10240
10230      T4=T4+T4
10235      GOTO 10255
10240      IF H5<>0 THEN 10255
10245      T2=(H8+T1)*(H8+T1)
10250      T4=T4+EXP(-T2)
10255      IF Z=0 THEN 10270
10260      K5=H9
10265      GOTO 10365
10270      Z1=Z2*T*SQR(2-T9)
10275      K1=K8-Z1
10280      IF K1<=-A THEN 10300
10285      Erfc=FNErfc(K1)
10290      K5=Erfc
10295      GOTO 10315

```

```

10300      Z=1
10305      K5=H9
10310      GOTO 10365
10315      IF Z3=0 THEN 10330
10320      K5=K5-K9
10325      GOTO 10365
10330      K1=K8+Z1
10335      IF K1>=A THEN 10355
10340      Erfc=FN Erfc(K1)
10345      K5=K5-Erfc
10350      GOTO 10365
10355      K5=K5-K9
10360      Z3=1
10365      G3=G3+K5*T4*T*Y(J+ABS(I))
10370      NEXT I
10375      E11cv=Y*G3
10380      IF E11cv>1 THEN RETURN 1
10385      RETURN E11cv
10390      FOR I=-N1 TO N1
10395          IF I=0 THEN 10370
10400          T=E3*(SGN(I)*X(J+ABS(I))+1)+D1
10405          T9=T*T
10410          T1=R8*(1-T9)
10415          T2=(H8-T1)*(H8-T1)
10420          T4=EXP(-T2)
10425          IF H5<>0 THEN 10440
10430          T2=(H8+T1)*(H8+T1)
10435          T4=T4+EXP(-T2)
10440          IF Z=0 THEN 10455
10445          K5=H9
10450          GOTO 10365
10455          K1=-Z2*T*SQR(2-T9)
10460          IF K1<=-A THEN 10480
10465          Erfc=FN Erfc(-K1)      !FN Erfc(K1)
10470          K5=2*Erfc      !2*(Erfc-1)
10475          GOTO 10365
10480          Z=1
10485          K5=H9
10490          GOTO 10365
10495      FNEND
10500      ! *****ERFC(X)*****
10505      DEF FN Erfc(X)
10510          K6=X*X
10515          IF ABS(X)>.5 THEN 10525
10520          RETURN .5-X*((( -3.56098437018E-2*K6+6.99638348862)*K6+21.9792616183)*K6+
242.667955231)/(((K6+15.0827976304)*K6+91.1649054045)*K6+215.05887587)+.5
10525          K4=ABS(X)
10530          IF K4>4 THEN 10550
10535          Erfc=((( -6.08581519597E-6*K4+.564371606864)*K4+4.26772010709)*K4+14.571
8985969)*K4+26.0947469561)*K4+22.8989928517
10540          Erfc=EXP(-K6)*Erfc/(((K4+7.56884822936)*K4+26.2887957588)*K4+50.273202
8638)*K4+51.9335706876)*K4+22.8989857499)

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10545 GOTO 10580
10550 IF X<-4.89 THEN RETURN 2
10555 IF X>=14.9891301 THEN RETURN 0
10560 K6=1/K6
10565 B1=.564189583548! 1/SQR(PI)
10570 Erfc=(B1-K6*(((3.24319519278E-2*K6+.243911029489)*K6+.119903955268)*K6+.
012130827639)/(((K6+1.43771227937)*K6+.489552441961)*K6+4.30026643453E-2))/K4
10575 Erfc=EXP(-X*X)*Erfc
10580 IF X<0 THEN RETURN 2-Erfc
10585 RETURN Erfc
10590 FNEND
10595 ! *****CIRCV*****
10600 ! THIS SUBROUTINE IS CALLED CIRCV. INPUT: R,D,S,V.
10605 ! IF V#0. OUTPUT: Circv=CIRCULAR COVERAGE FUNCTION. Circv GIVES THE
10606 ! PROBABILITY OF A SHOT FALLING, UNDER A NORMAL DISTRIBUTION WITH
10610 ! MEAN (0,0) AND EQUAL STANDARD DEVIATIONS, S, IN A CIRCLE OF RADIUS R
10611 ! OFFSET A DISTANCE D FROM (0,0).
10615 ! IF V=0, OUTPUT: Circv=GENERALIZED CIRCULAR ERROR FUNCTON. Circv GIVES
10616 ! THE PROBABILITY OF A SHOT FALLING, UNDER A NORMAL DISTRIBUTION WITH
10620 ! MEAN (0,0) AND STANDARD DEVIATIONS S1 AND S, IN A CIRCLE OF RADIUS R
10621 ! CENTERED AT (0,0). D = S1/S <= 1.
10625 ! FOR V=0, WITH S1 = 0, S1#S, Circv=ERF(R/(SQR(2)*S)).
10630 ! SUBROUTINES NEEDED: ERFC2, ERF.
10635 ! THE PARIAL DERIVATIVE OF Circv WITH RESPECT TO R IS CONTAINED WITHIN THE
10636 ! ROUTINE AND CAN BE EXTRACTED IF DESIRED.
10640 ! SOURCES: MATH OF COMP APRIL 1961, PP169,173 AND OCT.1961, PP 375,382.
10641 ! NWL REPORT #1768, JAN. 1962. NSWC REPORT#83-13, NOV. 1982.
10645 ! IEEE TRANS. INFO. TH. APRIL 1965, P. 312.
10650 ! PROGRAM IS SET FOR APPROXIMATELY EIGHT DECIMAL DIGIT ACCURACY.
10655 DEF FNCircv(R,D,S,V)
10660 IF R=0 THEN RETURN 0
10665 B1=.564189583548! 1/SQR(PI)
10670 C7=.707106781187
10675 C1=1E-8
10680 R1=R/S
10685 IF V<>0 THEN 10740
10690 IF ABS(D-.5)<=.5 THEN 10700
10695 PRINT "ERROR: D>1 OR D<0, IN CIRCV."
10700 D1=D
10705 R1=R/S
10710 IF D1<>0 THEN 10720
10715 RETURN FNERf(R1*C7) ! V=0 AND D=S1=0.
10720 T=.5*R1/D1
10725 R1=T*(1+D1)
10730 D1=T*(1-D1)
10735 GOTO 10745
10740 D1=D/S
10745 A1=R1-D1
10750 K1=ABS(A1)
10755 IF K1<6.02738 THEN 10770 !5.6123
10760 IF A1<0 THEN RETURN 0
10765 RETURN 1

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10770 T=R1*D1
10775 T3=.5*R1*R1
10780 B=.5*D1*D1
10785 N=0
10790 IF T>7 THEN 10930
10795 T1=C7*T-1
10800 T2=T3*B
10805 S0=EXP(-T3-B)
10810 S1=EXP(-B)
10815 IF T3>.0005 THEN 10830
10820 S1=S1*T3*(.5+((-T3/24+1/6)*T3-.5)*T3+.5)
10825 GOTO 10835
10830 S1=S1-S0
10835 S2=S0
10840 T0=S1
10845 N=N+1
10850 M=1/N
10855 S0=T2*M*M*S0
10860 T0=B*M*T0-S0
10865 S1=S1+T0
10870 S2=S2+S0
10875 IF T1>N THEN 10845
10880 IF T0>C1 THEN 10895
10885 P=S1
10890 GOTO 11080
10895 N=N+1
10900 M=1/N
10905 S0=T2*M*M*S0
10910 T0=B*M*T0-S0
10915 S1=S1+T0
10920 S2=S2+S0
10925 GOTO 10880
10930 T1=2*ABS(T3-B)
10935 K1=K1*C7
10940 T3=1/(T+T)
10945 T2=SQR(T3)
10950 S1=.5*A1*A1
10955 S2=EXP(-S1)
10960 S0=B1*T2*S2
10965 T0=(R1+D1)*C7*T2*FNerfc2(K1,S2)
10970 T2=S1*T3
10975 T3=.5*T3
10980 S1=T0
10985 S2=S0
10990 N=N+2
10995 M=N-1
11000 K1=M/N
11005 S0=K1*T3*S0
11010 T0=T1*S0-T2*K1*T0
11015 S0=M*S0
11020 S1=S1+T0
11025 S2=S2+S0
11030 IF T0-C1>0 THEN 10990
11035 IF S0-C1<=0 THEN 11075
11040 T1=ABS(S0)

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11045 N=N+2
11050 M=M-1
11055 S0=M*M*T3*S0/N
11060 IF ABS(S0)>=T1 THEN 11075
11065 S2=S2+S0
11070 GOTO 11035
11075 P=.5*ABS(1+SGN(A1)-S2-SGN(A1)*S1)
11080 IF V<>0 THEN 11090
11085 P=ABS(P+P+S2-1)
11090 IF P>1 THEN RETURN 1
11095 RETURN P !***EXIT***
11100 FNEND
11105 ! *****ERFC2*****
11110 ! Erfc=ERFC(X). Exp=EXP(-X*X) IF AVAILABLE, OTHERWISE Exp=-1.
11115 DEF FNERfc2(X,Exp)
11120 K6=X*X
11125 IF ABS(X)>.5 THEN 11135
11130 RETURN .5-X*(((3.56098437018E-2*K6+6.99638348862)*K6+21.9792616183)*K6+
242.667955231)/(((K6+15.0827976304)*K6+91.1649054045)*K6+215.05887587)+.5
11135 K4=ABS(X)
11140 IF K4>4 THEN 11165
11145 IF Exp<0 THEN Exp=EXP(-K6)
11150 Erfc=(((6.08581519597E-6*K4+.564371606864)*K4+4.26772010709)*K4+14.571
8985969)*K4+26.0947469561)*K4+22.8989928517
11155 Erfc=Exp*Erfc/(((K4+7.56884822936)*K4+26.2887957588)*K4+50.2732028638)
*K4+51.9335706876)*K4+22.8989857499)
11160 GOTO 11200
11165 IF X<-4.89 THEN RETURN 2
11170 IF X>=14.9891301 THEN RETURN 0
11175 IF Exp=-1 THEN Exp=EXP(-K6)
11180 K6=1/K6
11185 B1=.564189583548! 1/SQR(PI)
11190 Erfc=(B1-K6*(((3.24319519278E-2*K6+.243911029489)*K6+.119903955268)*K6+.
012130827639)/((K6+1.43771227937)*K6+.489552441961)*K6+4.30026643453E-2))/K4
11195 Erfc=Exp*Erfc
11200 IF X<0 THEN RETURN 2-Erfc
11205 RETURN Erfc
11210 FNEND
11215 ! *****GQUAD*****
11220 ! Gquad IS A QUADRATURE ROUTINE WHICH USES GAUSSIAN MULTIPLIERS TO OBTAIN
11221 ! THE INTEGRAL I FROM A TO B OF Fh(T), (SEE NEXT SUBPROGRAM.)
11225 ! A,B : ARE LOWER AND UPPER LIMITS OF INTEGRATION.
11230 ! Ierr : IF 0 THEN ANSWER O.K. IF 1 THEN MAXIMUM NO. OF SUBDIVISIONS USED.
11235 ! M : MAXIMUM NO. OF EQUAL SUBDIVISIONS OF [A,B] DESIRED WITH GAUSSIAN
11236 ! APPLIED ON EACH SUBDIVISION.
11240 ! I1 : VALUE OF INTEGRAL STORED IN I1.
11245 ! E : |I1-I0| < E SAYS DIFFERENCE BETWEEN TWO CONSECUTIVE CALCULATIONS FOR
11246 ! I ARE LESS THAN E.
11250 ! N : 2N = ORDER OF GAUSSIAN MULTIPLIERS USED. ONE MAY SPECIFY N=3,4,6,8,
11251 ! 10,12. ONLY THESE VALUES OF N ALLOWED.
11255 ! X(*),Y(*) : STORED VALUES OF GAUSSIAN ABSCISSAS AND MULTIPLIERS ON
11256 ! [-1,1].
11260 ! J : NO. OF EQUAL SUBDIVISIONS USED. J <=M.

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11265 ! NEEDS X(*), Y(*) ARRAYS WHICH ARE STORED IN "GAUSS"; FUNCTION FN.
11270 SUB Gquad(A,B,Ierr,M,I1,E,N,N2,R3,H,K,L,S1,S2,S3,V,D,W9)
11275   COM X(*),Y(*)
11280   J1=Ierr=0
11285   IF N=3 THEN 11355
11290   J1=3
11295   IF N=4 THEN 11355
11300   J1=7
11305   IF N=6 THEN 11355
11310   J1=13
11315   IF N=8 THEN 11355
11320   J1=21
11325   IF N=10 THEN 11355
11330   J1=31
11335   IF N=12 THEN 11355
11340   PRINT "N AND J1 NOT RIGHT"
11345   BEEP
11350   STOP
11355   D3=B-A
11360   J=0
11365   I0=1E50
11370   J=J+1
11375   E1=A
11380   G=0
11385   D1=D3/J
11390   D2=D1/2
11395   K1=0
11400   K1=K1+1
11405   E0=E1
11410   E1=E0+D1
11415   E2=(E1+E0)*.5
11420   FOR I=-N TO N
11425     IF I=0 THEN 11445
11430     T=D2*(SGN(I)*X(J1+ABS(I)))+E2
11435     F=Y(J1+ABS(I))*FNFn(T,R3,H,K,L,S1,S2,S3,V,D,W9)
11440     G=G+F
11445   NEXT I
11450   IF K1<>J THEN 11400
11455   I1=D2*G
11460   IF ABS(I1-I0)<E THEN 11480
11465   I0=I1
11470   IF J<>M THEN 11370
11475   Ierr=1
11480   N2=N*J*(J+1)!NO. OF FCN EVALUATIONS.
11485 SUBEND
11490 ! *****FN*****
11495 ! NEEDED FOR GQUAD. NEEDS ELLCV, CIRCV.
11500 DEF FNFn(T5,R3,H,K,L,S1,S2,S3,V,D,W9)
11505   COM X(*),Y(*)
11510   C8=1.41421356237
11515   J1=0
11520   P=L-C8*S3*T5
11525   J1=EXP(-T5*T5)
11530   IF L<>0 THEN 11545
11535   J1=J1+J1

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11540 GOTO 11565
11545 IF W9<>0 THEN 11565
11550 V2=2*L*P/(S3*S3)
11555 IF V2>27 THEN 11565
11560 J1=J1*(.5+EXP(-V2)+.5)
11565 R=SQR(R3*R3-P*P)
11570 IF V=2 THEN 11585
11575 P=FNCircu(R,D,S1,V)
11580 GOTO 11590
11585 P=FNEllcu(R,H,K,S1,S2)
11590 J1=J1*P
11595 RETURN J1
11600 FNEND
11605 ! *****ERF3*****
11610 ! Erf=ERF(X). IF AVAILABLE Exp=EXP(-X*X) ELSE Exp=-1.
11615 DEF FNErf3(X,Exp)
11620 X2=X*X
11625 IF ABS(X)>.5 THEN 11640
11630 Erf=X*((( -3.56098437018E-2*X2+6.99638348862)*X2+21.9792616183)*X2+242.66
7955231)/(((X2+15.0827976304)*X2+91.1649054045)*X2+215.05887587)
11635 RETURN Erf
11640 K4=ABS(X)
11645 IF K4>4 THEN 11680
11650 K3=((( -6.08581519597E-6*K4+.564371606864)*K4+4.26772010709)*K4+14.57189
85969)*K4+26.0947469561)*K4+22.8989928517
11655 K3=K3/(((K4+7.56884822936)*K4+26.2887957588)*K4+50.2732028638)*K4+51.9
335706876)*K4+22.8989857499)
11660 IF Exp<0 THEN Exp=EXP(-X2)
11665 Erf=1-Exp*K3
11670 IF X<0 THEN RETURN -Erf
11675 RETURN Erf
11680 Erf=1
11685 IF K4>=4.883 THEN 11715
11690 K6=1/X2
11695 B1=.564189583548! 1/SQR(P1)
11700 Erf=(B1-K6*(((3.24319519278E-2*K6+.243911029489)*K6+.119903955268)*K6+.0
12130827639)/((K6+1.43771227937)*K6+.489552441961)*K6+4.30026643453E-2))/K4
11705 IF Exp<0 THEN Exp=EXP(-X2)
11710 Erf=1-Exp*Erf
11715 IF X<0 THEN RETURN -Erf
11720 RETURN Erf
11725 FNEND
11730 ! *****ELLRC*****
11735 ! FINDS LENGTH OF SIDE OF CUBE WITH CENTER (H,K,L) WHICH CONTAINS P3 OF
11740 ! THE NORMAL DISTRIBUTION WITH MEAN 0 AND STANDARD DEVIATIONS (S1,S2,S3).
11745 ! FOR FNEllrc CHECK T6. IF T6>40 RESULT SUSPECT.
11750 ! SUBROUTINES NEEDED: ERF, ERF3, ERF3.
11755 DEF FNEllrc(H,K,L,S1,S2,S3,P3,Rmin,Rmax,T6)
11760 T6=0
11765 IF (P3>1) OR (P3<0) THEN RETURN -1E99
11770 IF P3=1 THEN RETURN 1E99
11775 IF P3=0 THEN RETURN 0
11780 Sq=1.41421356237
11785 U1=1/(Sq*S1)

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11790  U2=1/(Sq*S2)
11795  Sqpi=2.50662827463
11800  U3=1/(Sq*S3)
11805  Y1=1/(Sqpi*S1)
11810  Y2=1/(Sqpi*S2)
11815  Y3=1/(Sqpi*S3)
11820  R1=Rmax
11825  R0=Rmin
11830  T6=0
11835  A=.5*(R1+R0)
11840  GOSUB 11930
11845  IF (ABS(U)<.1*P3) OR (ABS(R1-R0)<.01*R0) THEN 11865
11850  GOTO 11835
11855  A=.5*(R1+R0)
11860  GOSUB 11930
11865  IF C0=0 THEN C0=EXP(-2*H*A/(S1*S1))
11870  IF C2=0 THEN C2=EXP(-2*K*A/(S2*S2))
11875  IF C4=0 THEN C4=EXP(-2*L*A/(S3*S3))
11880  B1=Y1*C1*(1+C0)*F2*F3+Y2*C3*(1+C2)*F1*F3+Y3*C5*(1+C4)*F1*F2
11885  IF B1<=0 THEN 11835
11890  A1=A-U/B1
11895  IF ABS(F-P3)<1E-10 THEN RETURN A1
11900  IF (ABS(A1-A)<5E-5*A1) AND (ABS(F-P3)<1E-3*P3) THEN RETURN A1
11905  IF ABS(A1-A)<=1E-8 THEN RETURN A1
11910  A=A1
11915  IF T6>40 THEN RETURN A1
11920  IF (A<R0) OR (A>R1) THEN 11855
11925  GOTO 11860
11930  T0=(H+A)*U1
11935  T1=(H-A)*U1
11940  T2=(K+A)*U2
11945  T3=(K-A)*U2
11950  T4=(L+A)*U3
11955  T5=(L-A)*U3
11960  C1=F=EXP(-T1*T1)
11965  C3=F=EXP(-T3*T3)
11970  C5=F=EXP(-T5*T5)
11975  C0=C2=C4=0
11980  IF T1<0 THEN 12005
11985  IF C1=0 THEN GOSUB 12085
11990  C0=EXP(-2*H*A/(S1*S1))
11995  F1=.5*C1*(FNERfc1(T1,1)-C0*FNERfc1(T0,1))
12000  GOTO 12010
12005  F1=.5*(FNERfc1(T0)-FNERfc3(T1,C1))
12010  IF T3<0 THEN 12035
12015  IF C3=0 THEN GOSUB 12085
12020  C2=EXP(-2*K*A/(S2*S2))
12025  F2=.5*C3*(FNERfc1(T3,1)-C2*FNERfc1(T2,1))
12030  GOTO 12040
12035  F2=.5*(FNERfc1(T2)-FNERfc3(T3,C3))
12040  IF T5<0 THEN 12065
12045  IF C5=0 THEN GOSUB 12085
12050  C4=EXP(-2*L*A/(S3*S3))
12055  F3=.5*C5*(FNERfc1(T5,1)-C4*FNERfc1(T4,1))

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12060 GOTO 12070
12065 F3=.5*(FNErf(T4)-FNErf3(T5,C5))
12070 F=F1*F2*F3
12075 U=F-P3
12080 IF U>0 THEN 12095
12085 R0=A
12090 GOTO 12100
12095 R1=A
12100 T6=T6+1
12105 IF ABS(F-.5)=.5 THEN 11835
12110 RETURN
12115 FNEND
12120 ! *****FNXerfc(X) = X*EXP(X*X)*ERFC(X)*****
12125 DEF FNXerfc(X)
12130 X2=X*X
12135 IF ABS(X)>.5 THEN 12150
12140 Erfc1=.5-X*((( -3.56098437018E-2*X2+6.99638348862)*X2+21.9792616183)*X2+2
42.667955231)/(((X2+15.0827976304)*X2+91.1649054045)*X2+215.05887587)+.5
12145 RETURN X*EXP(X2)*Erfc1
12150 K4=ABS(X)
12155 IF K4>4 THEN 12180
12160 K3=((( -6.08581519597E-6*K4+.564371606864)*K4+4.26772010709)*K4+14.57189
85969)*K4+26.0947469561)*K4+22.8989928517
12165 Erfc1=K3/(((K4+7.56884822936)*K4+26.2887957588)*K4+50.2732028638)*K4+5
1.9335706876)*K4+22.8989857499)
12170 IF X<0 THEN RETURN X*(2*EXP(X2)-Erfc1)
12175 RETURN X*Erfc1
12180 IF X<-4.900 THEN RETURN 2*X*EXP(X*X)
12185 K6=1/X2
12190 B1=.564189583548! 1/SQR(PI)
12195 Erfc1=(B1-K6*(((3.24319519278E-2*K6+.243911029489)*K6+.119903955268)*K6+
.012130827639)/(((K6+1.43771227937)*K6+.489552441961)*K6+4.30026643453E-2))/K4
12200 IF X<0 THEN RETURN X*(2*EXP(X2)-Erfc1)
12205 RETURN X*Erfc1
12210 FNEND
12215 ! *****FNDxerf3(X,Exp) = ERF(X)/X; EXP(-X*X) AVAILABLE IF Exp > 0.**
12220 DEF FNDxerf3(X,Exp)
12225 X2=X*X
12230 IF ABS(X)>.5 THEN 12245
12235 Erf=((( -3.56098437018E-2*X2+6.99638348862)*X2+21.9792616183)*X2+242.6679
55231)/(((X2+15.0827976304)*X2+91.1649054045)*X2+215.05887587)
12240 RETURN Erf
12245 K4=ABS(X)
12250 IF K4>4 THEN 12275
12255 K3=((( -6.08581519597E-6*K4+.564371606864)*K4+4.26772010709)*K4+14.57189
85969)*K4+26.0947469561)*K4+22.8989928517
12260 K3=K3/(((K4+7.56884822936)*K4+26.2887957588)*K4+50.2732028638)*K4+51.9
335706876)*K4+22.8989857499)
12265 IF Exp<0 THEN Exp=EXP(-X2)
12270 RETURN (1-Exp*K3)/K4

```

```

12275 IF K4>=4.883 THEN RETURN 1/K4
12280 K6=1/X2
12285 B1=.564189583548! 1/SQR(PI)
12290 Erf=(B1-K6*(((3.24319519278E-2*K6+.243911029489)*K6+.119903955268)*K6+.0
12130827639)/(((K6+1.43771227937)*K6+.489552441961)*K6+4.30026643453E-2))/K4
12295 IF Exp<0 THEN Exp=EXP(-X2)
12300 RETURN (1-Exp*Erf)/K4
12305 FNEND
12310 ! *****DXDAWS(X)*****
12315 ! Daws(X) = 1/X*EXP(-X*X)* INTEGRAL FROM 0 TO X OF EXP(T*T) DT.
12316 ! REF: MATH OF COMP. 1970, PP. 171-178. DAWSON'S INTEGRAL/X
12320 DEF FNDxdaws(K1)
12325 Daws=K1*K1
12330 IF ABS(K1)>2.5 THEN 12350
12335 K4=((((-2.38594565696E-2*Daws+1.67795116189)*Daws-15.1982152422)*Daws+41
9.67290228)*Daws-1284.05832279)*Daws+10832.6558873
12340 Daws=K4/((((Daws+19.9422336364)*Daws+219.728331833)*Daws+1489.43557242)
*Daws+5937.71276935)*Daws+10832.6558772)
12345 RETURN Daws
12350 IF ABS(K1)>3.5 THEN 12380
12355 K4=88.7619386764/(Daws+3.47393742586)
12360 K4=13.8216341182/(Daws-12.6817901598+K4)
12365 K4=37.3902763512/(Daws+4.07068101667+K4)
12370 Daws=(.214221965778/(Daws-4.91605365741+K4)+.500652754437)/Daws
12375 RETURN Daws
12380 IF ABS(K1)>=5 THEN 12405
12385 K4=17.761778077/(Daws-4.1083233770)
12390 K4=-.53714272981/(Daws-10.385024821+K4)
12395 Daws=(.24924632421/(Daws-1.5853935006+K4)+.50000965199)/Daws
12400 RETURN Daws
12405 K4=-4.12544065608/(Daws-11.1952164237)
12410 K4=-2.48787658804/(Daws-4.67312022141+K4)
12415 Daws=.5*(1+(.749999190567/(Daws-2.50017116686+K4)+.500000001674)/Daws)/D
aws
12420 RETURN Daws
12425 FNEND

```



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